Lecture XXVI : § 5.8 Operations with linear transformations

TODAY'S GOAL: Describe operations between linear transformations $T : V \to W$

Correspondence to keep in mind: $\boxed{T : \mathbb{R}^n \to \mathbb{R}^m \text{ linear} \Longleftrightarrow A \ m\times n \text{ matrix}}$

§1 Summary:

- $M_{m\times n}$, $\{T : \mathbb{R}^n \to \mathbb{R}^m : T \text{ linear}\}$ & $\{T : V \to W : T \text{ linear}\}$ are vector spaces (addition & scalar mult)
- We have multiplication / composition for some pairs of matrices / linear transformations

Operation	Matrices	$T : \mathbb{R}^n \to \mathbb{R}^m$ linear	$T : V \to W$ linear
(I) Addition	$A+C$ matrix $(A+C)_{ij} = A_{ij} + C_{ij}$	$F+G : \mathbb{R}^n \to \mathbb{R}^m$ linear $(F+G)_{(\vec{v})} = F_{(\vec{v})} + G_{(\vec{v})}$	$F+G : V \to W$ linear $(F+G)_{(\vec{v})} = F_{(\vec{v})} + G_{(\vec{v})}$
(II) Scalar Multiplication	$\alpha . A$ matrix $(\alpha A)_{ij} = \alpha A_{ij}$	$\alpha . T : \mathbb{R}^n \to \mathbb{R}^m$ $(\alpha . T)_{(\vec{v})} = \alpha T_{(\vec{v})}$	$\alpha . T : \mathbb{R}^n \to \mathbb{R}^m$ $(\alpha . T)_{(\vec{v})} = \alpha T_{(\vec{v})}$
(III) Multiplication vs Composition	A in $M_{m\times n}$ C in $M_{s\times m}$ Then CA in $M_{s\times n}$	$F : \mathbb{R}^n \to \mathbb{R}^m$ linear $G : \mathbb{R}^m \to \mathbb{R}^s$ linear Then $G\circ F : \mathbb{R}^n \to \mathbb{R}^s$ linear $G\circ F(\vec{v}) = G(\underbrace{F(\vec{v})}_{\text{in } \mathbb{R}^m})$	$F : V \to W$ linear $G : W \to U$ linear Then $G\circ F : V \to U$ linear $G\circ F(\vec{v}) = G(\underbrace{F(\vec{v})}_{\text{in } W})$

$\boxed{\begin{matrix} F \text{ has matrix } A \\ G \text{ — } C \end{matrix}}$ (we saw this in Lecture 20)

§2 Examples: We show examples & highlight properties

EXAMPLE 1 $T_1 : \mathcal{P}_2 \to \mathbb{R}$, $P(x) \mapsto P(1)$, $T_2 : \mathcal{P}_2 \to \mathbb{R}$, $P(x) \mapsto P'(2)$ both are linear transf.

$T_1(a+bx+cx^2) = a+b+c$

$T_2(a+bx+cx^2) = b+2c(2) = b+4c$

(linear expressions in $[P]_{\{1,x,x^2\}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$)

NEW FUNCTION: (by addition)

- $T_1 + T_2 : \mathcal{P}_2 \to \mathbb{R}$ is also linear
  $P(x) \longmapsto P(1) + P'(2)$

In fact: $(T_1+T_2)(a+bx+cx^2) = a+b+c+b+4c = a+2b+5c$

- $\mathcal{N}(T_1+T_2) = \{a+bx+cx^2 : a+2b+5c=0\}$

$$= \{(-2b-5c)+bx+cx^2 : b,c \in \mathbb{R}\} = \{b(-2+x)+c(-5+x^2)\}$$
$$= Sp(-2+x,\ -5+x^2) \rightsquigarrow \dim = 2$$

- $\mathcal{N}(T_1) = \{a+bx+cx^2 : a+b+c=0\} = Sp((-1+x), (-1+x^2))$
- $\mathcal{N}(T_2) = \{a+bx+cx^2 : b+4c=0\} = Sp(1, -4+x^2)$

Observation: In general, we should **NOT** expect any relation between $\mathcal{N}(T_1)$, $\mathcal{N}(T_2)$ & $\mathcal{N}(T_1+T_2)$

- $\mathcal{R}(T_1+T_2)$ has dimension $= \dim \mathcal{P}_2 - \text{nullity}(T_1+T_2) = 3-2 = 1$

Since $\mathcal{R}(T_1+T_2)$ is a subspace of $\mathbb{R}$ & both have the same dimension, we conclude $\mathcal{R}(T_1+T_2) = \mathbb{R}$

NEW FUNCTION: (by scalar multiplication)

$$3T_1 : \mathcal{P}_2 \longrightarrow \mathbb{R} \quad \text{is linear} \qquad (3T_1)(a+bx+cx^2) = 3a+3b+3c$$
$$p \longmapsto 3p(1)$$

$$\mathcal{N}(3T_1) = \mathcal{N}(T) \quad \& \quad \mathcal{R}(3T_1) = \mathcal{R}(T_1)$$

Observation 2: In general: $\mathcal{N}(\alpha T) = \mathcal{N}(T)$ as long as $\underline{\underline{\alpha \neq 0}}$

$T: \mathbb{V} \to \mathbb{W}$ linear $\qquad \mathcal{R}(\alpha T) = \mathcal{R}(T)$

For $\underline{\underline{\alpha = 0}}$ we get $\mathcal{N}(0 \cdot T) = \mathbb{V}$ & $\mathcal{R}(0 \cdot T) = \{0_{\mathbb{W}}\}$

EXAMPLE 2

- $T_1 : M_{2\times 3} \longrightarrow \mathcal{P}_3$ linear

$$A \longmapsto a_{11}x^3 + (a_{12}-a_{13})x^2 + a_{23}$$

- $T_2 : \mathcal{P}_3 \longrightarrow \mathbb{R}^2$ is linear

$$a+bx+cx^2+dx^3 \longmapsto \begin{bmatrix} d-c \\ d-b+a \end{bmatrix}$$

NEW FUNCTION: T_2 composed with T_1

$$T_2 \circ T_1 : M_{2\times 3} \xrightarrow{T_2} \mathcal{P}_3 \xrightarrow{T_1} \mathbb{R}^2$$

$$A \longmapsto \underbrace{\boxed{a_{11}}}_{=d}x^3 + \underbrace{\boxed{(a_{12}-a_{13})}}_{=c}x^2 + \underbrace{\boxed{a_{23}}}_{=a}\ (b=0) \longmapsto \begin{bmatrix} a_{11}-(a_{12}-a_{13}) \\ a_{11} - 0 + a_{23} \end{bmatrix} = \begin{bmatrix} a_{11}-a_{12}+a_{13} \\ a_{11}+a_{23} \end{bmatrix}$$

linear in entries of A

In conclusion: $T_2\circ T_1(A) = \begin{bmatrix} a_{11}-a_{12}+a_{13} \\ a_{11}+a_{23}\end{bmatrix}$

- **Natural question 1:** Are $\mathcal{N}(T_1)$ & $\mathcal{N}(T_2\circ T_1)$ related? Both are subspaces of $M_{2\times 3}$

Answer: YES, they are!

$\vec v$ in $\mathcal N(T_1)$, then $T_1(\vec v)=\mathbb{0}$ in $\mathcal P_2$. Now apply T_2

$$(T_2\circ T_1)(\vec v) = T_2(T_1(\vec v)) = T_2(\mathbb{0}_{\mathcal P_2}) \underset{T_2 \text{ linear so } \mathbb{0}\to\mathbb{0}.}{=} \begin{bmatrix}0\\0\end{bmatrix}$$

conclude: $\vec v$ in $\mathcal N(T_1\circ T_2)$

$\boxed{\text{Observation 3:}}$ If $T_1: V\to W$, $T_2: W\to U$ linear, then all $\vec v$ in $\mathcal N(T_1)$ also lie in $\mathcal N(T_2\circ T_1)$. In symbols: $\boxed{\mathcal N(T_1)\subseteq \mathcal N(T_2\circ T_1)}$

- **Natural question 2:** Are $\mathcal R(T_2)$ & $\mathcal R(T_2\circ T_1)$ related? Both are subspaces of $\mathbb R^2$

Pick $\vec w$ in $\mathcal R(T_2\circ T_1)$. Then $\vec w = T_2\circ T_1(\vec v)$ for some $\vec v$ in $M_{2\times 3}$

$$= T_2(\underbrace{T_1(\vec v)}_{\vec u \text{ in } \mathcal P_3}) \rightsquigarrow \text{so } \vec w \text{ in } \mathcal R(T_2)$$

- Can we say something else?

$\mathcal R(T_2) = ?$

$$\mathcal R(T_2) = Sp(T_2(1), T_2(x), T_2(x^2), T_2(x^3)) = Sp\left(\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\-1\end{bmatrix}, \begin{bmatrix}-1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right) = \mathbb R^2$$

$\mathcal R(\underbrace{T_2\circ T_1}_{=:T}) = ?$

$$\mathcal R(T_2\circ T_1) = Sp(T(E_{11}), T(E_{12}), T(E_{13}), T(E_{21}), T(E_{22}), T(E_{23})) = Sp\left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right) = \mathbb R^2$$

In this case: $\mathcal R(T_2) = \mathcal R(T) = \mathbb R^2$

$\boxed{\text{Observation 4:}}$ If $T_1: V\to W$, $T_2: W\to U$ linear, then all vectors in $\mathcal R(T_2\circ T_1)$ also lie in $\mathcal R(T_2)$. In symbols: $\boxed{\mathcal R(T_2\circ T_1)\subseteq \mathcal R(T_2)}$

Furthermore, if $\mathcal R(T_1) = W$, then $\mathcal R(T_2\circ T_1) = \mathcal R(T_2)$

- Next, we study surjective & invertible transformations:

§3. Surjective or Onto Transformations:

Def A linear transformation $T: V \longrightarrow W$ is onto (or surjective) if $R(T) = W$

Example 1: $T_2: P_3 \longrightarrow \mathbb{R}^2$ (from Example 2) $T_2(a+bx+cx^2+dx^3) = \begin{bmatrix} d-c \\ d-b+a \end{bmatrix}$ is onto since $R(T_2) = \mathbb{R}^2$

Example 2: $T: P_3 \longrightarrow P_2$ $T(P(x)) = P'(x)$ is onto.

Why? Fix $f(x) = a+bx+cx^2$ in P_2 we pick $P(x) = \int_0^x f(t)\,dt = at + \frac{b t^2}{2} + \frac{c t^3}{3}\Big|_0^x$
$= ax + \frac{b}{2}x^2 + \frac{cx^3}{3}$

Then $T(P(x)) = f(x)$ by Fundamental Thm of Calculus

Alternative way to check this? $R(P_3) = \mathrm{Sp}(\underbrace{T(1)}_{=0}, \underbrace{T(x)}_{=1}, \underbrace{T(x^2)}_{=2x}, \underbrace{T(x^3)}_{=3x^2})$
$= \mathrm{Sp}(1, 2x, 3x^2) = \mathrm{Sp}(1, x, x^2) = P_2$

Example 3: $\tilde{T}: P_3 \longrightarrow P_3$ $\tilde{T}(P(x)) = P'(x)$ is NOT onto because $R(\tilde{T}) = P_2$
(same formula as T from Example 2) $\quad = R(T)$

Q: What can we use to check if $R(T) = W$? A: Dimension!

Proposition: If $T: V \longrightarrow W$ is linear & dimension of W is finite, then we can check if T is onto by checking if $\dim R(T) = \dim W$.

§4. Invertible Transformations

This notion combines both the notion of invertible matrices & maps $f: \mathbb{R} \longrightarrow \mathbb{R}$ that are bijections.

Def: A linear transformation $T: V \longrightarrow W$ is invertible if we can find another linear transformation $S: W \longrightarrow V$ satisfying:

(1) $S \circ T: V \longrightarrow V$, $\vec{v} \longmapsto \vec{v}$ & (2) $T \circ S: W \longrightarrow W$, $\vec{w} \longmapsto \vec{w}$.

(so $S \circ T =$ identity map on V) (so $T \circ S =$ identity map on W)

Observation: If T is invertible, we can only find one S with these properties.

Because of this, we can write $S = T^{-1}$. (like we did when finding inverses of (square) non-singular matrices)

Alternative name: An invertible linear transformation is also called an isomorphism.

Why? It basically says that for all practical purposes V & W can be thought of as the same vector space (E.g. $T: \mathcal{P}_2 \longrightarrow \mathbb{R}^3$ is an isomorphism)
$$a+bx+cx^2 \longmapsto \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Special case: $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ linear map associated to an $m \times n$ matrix A
$$\vec{v} \longmapsto A\vec{v}$$

Then: T is invertible if and only if $m = n$ & A is an invertible matrix

Furthermore: $T^{-1}: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the linear map associated to A^{-1}.
$$\vec{w} \longmapsto A^{-1}\vec{w}$$

Why? (1) $T^{-1} \circ T(\vec{v}) = T^{-1}(A\vec{v}) = A^{-1}(A\vec{v}) = (A^{-1}A)\vec{v} = \mathrm{Id}\,\vec{v} = \vec{v}$ for all $\vec{v}$

(2) $T \circ T^{-1}(\vec{w}) = T(A^{-1}\vec{w}) = A(A^{-1}\vec{w}) = (AA^{-1})\vec{w} = \mathrm{Id}\cdot\vec{w} = \vec{w}$ for all $\vec{w}$

Main Example Pick V a finite-dimensional vector space & fix a basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_p\}$ for V

Then:
$$\boxed{\begin{array}{c} T: V \longrightarrow \mathbb{R}^p \\ \vec{v} \longmapsto [\vec{v}]_{\mathcal{B}} \end{array}}$$
is an invertible linear transformation

Why? Propose a formula for an inverse!

Set $S: \mathbb{R}^p \longrightarrow V$ it is linear by construction.
$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix} \longmapsto \alpha_1\vec{v}_1 + \dots + \alpha_p\vec{v}_p$$

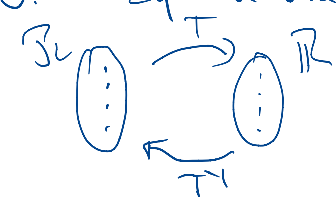
(1) $S \circ T(\vec{v}) = S\Big(\underbrace{\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix}}_{= [\vec{v}]_{\mathcal{B}}}\Big) = \alpha_1\vec{v}_1 + \dots + \alpha_p\vec{v}_p = \vec{v}$

(2) $T \circ S\left(\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix}\right) = T(\alpha_1\vec{v}_1 + \dots + \alpha_p\vec{v}_p) = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix}$ by definition of $[\ ]_{\mathcal{B}}$.

Conclude: $S = T^{-1}$

Non-examples (1) $T: \mathcal{P}_2 \longrightarrow \mathbb{R}$ is NOT invertible
$$P \longmapsto P(1)$$

Issue: 0 and $1-x$ both map to 0. If we could write T^{-1}, what should be the polynomial $T^{-1}(0)$?



In particular: $1-x = T^{-1} \circ T(1-x) = T^{-1}(\mathbb{0}) = \mathbb{0}$ but $1-x$ is <u>NOT</u> the $\mathbb{0}$ of $\mathcal{P}_2$!

(if T^{-1} exists; T^{-1} is linear)

Problem: $\mathcal{N}(T) \neq \{\mathbb{0}_V\}$ (Equivalently: T is <u>NOT</u> injective)

(2) $T: \mathbb{R} \longrightarrow \mathbb{R}^2$, $x \longmapsto \begin{bmatrix} 2x \\ x \end{bmatrix}$ is linear but <u>NOT</u> invertible

Why? $\mathcal{R}(T)$ = line through $(0,0)$ with direction $\binom{2}{1}$

So $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is not in $\mathcal{R}(T)$. What should be $\boxed{T^{-1}\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)}$? (in $\mathbb{R}$) Call this # by a

In particular $\begin{bmatrix} 3 \\ 1 \end{bmatrix} = T \circ T^{-1}\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = T(a) = \begin{bmatrix} 2a \\ a \end{bmatrix} = a \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(If T^{-1} exists)

<u>BuT</u>: $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ are not parallel, so we can never have a number a verifying $\begin{bmatrix} 3 \\ 1 \end{bmatrix} = a \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

<u>Conclude</u>: T^{-1} cannot exist.

<u>Problem</u>: $\mathcal{R}(T) \neq \mathbb{R}^2$ (T was not surjective)

These two last examples give insight into the following question:

<u>Q</u>: How can we determine if a linear transformation $T: V \longrightarrow W$ is invertible?

The next proposition gives some easy things to check:

<u>Proposition</u>: If $T: V \longrightarrow W$ linear is invertible then:

(1) $\mathcal{N}(T) = \{\mathbb{0}_V\}$ (T is injective)

(2) $\mathcal{R}(T) = W$ (T is surjective)

<u>Proof</u>: We saw the main ideas in the examples above. But more precisely:

(1) We know $T^{-1} \circ T: V \longrightarrow V$, $\vec{v} \longmapsto \vec{v}$ so $\mathcal{N}(T^{-1} \circ T) = \{\mathbb{0}_V\}$

BuT we know from $\boxed{\text{Observation 3}}$ that $\mathcal{N}(T) \subseteq \mathcal{N}(T^{-1} \circ T) = \{\mathbb{0}_V\}$

So we conclude $\mathcal{N}(T)$ can only contain $\mathbb{0}_V$.

(2) We know $T \circ T^{-1}: W \longrightarrow W$, $\vec{w} \longmapsto \vec{w}$ so $\mathcal{R}(T \circ T^{-1}) = W$

But we know from $\boxed{\text{Observation 4}}$ that $\underbrace{R(T\circ T^{-1})}_{=\mathbb{W}} \subseteq R(T)$

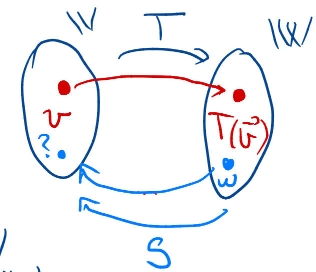
So all vectors $\vec{w}$ in $\mathbb{W}$ lie in $R(T)$. Since $R(T)$ is a subspace of $\mathbb{W}$ we conclude $R(T)=\mathbb{W}$. $\square$

- It turns out these 2 things are enough!

<u>Theorem 1</u>: If $T:\mathbb{V}\longrightarrow\mathbb{W}$ is linear with $N(T)=\{0_{\mathbb{V}}\}$ & $R(T)=\mathbb{W}$ then T is invertible.

<u>Proof</u>: We want to define the inverse to T:

$$S:\ \mathbb{W}\longrightarrow\mathbb{V}$$
$$\vec{w}\longmapsto ?$$



Once S is defined, we must check: S is linear & $\begin{cases} S\circ T(\vec{v})=\vec{v} & \text{for all } \vec{v} \text{ in } \mathbb{V}\\ T\circ S(\vec{w})=\vec{w} & \text{for all } \vec{w} \text{ in } \mathbb{W}\end{cases}$

- Since $R(T)=\mathbb{W}$, we can find $\vec{v}$ in $\mathbb{V}$ with $T(\vec{v})=\vec{w}$. But since $N(T)=\{0_{\mathbb{V}}\}$ there is no other $\vec{u}$ with $T(\vec{u})=\vec{w}$.

So, we define $S(\vec{w})=\vec{v}$ if $T(\vec{w})=\vec{v}$. (our hands are tied!)

<u>By design</u>: $S\circ T(\vec{v})=S(\underbrace{T(\vec{v})}_{=\vec{w}})=\vec{v}$ for all $\vec{v}$ in $\mathbb{V}$ ✓

by definition of S.

- <u>S is linear</u>:

(1) If $\vec{w}_1,\vec{w}_2$ in $\mathbb{W}$, pick $\vec{v}_1,\vec{v}_2$ in $\mathbb{V}$ with $T(\vec{v}_1)=\vec{w}_1 \rightsquigarrow S(\vec{w}_1)=\vec{v}_1$

$T(\vec{v}_2)=\vec{w}_2 \rightsquigarrow S(\vec{w}_2)=\vec{v}_2$

<u>Note</u> $T(\vec{v}_1+\vec{v}_2)=\vec{w}_1+\vec{w}_2 \rightsquigarrow S(\vec{w}_1+\vec{w}_2)=\vec{v}_1+\vec{v}_2$

<u>Conclude</u>: $S(\vec{w}_1+\vec{w}_2)=S(\vec{w}_1)+S(\vec{w}_2)$ ✓

(2) If $\vec{w}$ in $\mathbb{W}$ & α in $\mathbb{R}$, pick $\vec{v}$ in $\mathbb{V}$ with $T(\vec{v})=\vec{w}$ $\rightsquigarrow S(\vec{w})=\vec{v}$

Then $T(\alpha\cdot\vec{v})=\alpha T(\vec{v})=\alpha\vec{w} \rightsquigarrow S(\alpha\vec{w})=\alpha\vec{v})$

<u>Conclude</u>: $S(\alpha\vec{w})=\alpha S(\vec{w})$. ✓

- <u>Only missing thing to check</u>: $T\circ S(\vec{w})=\vec{w}$ for all $\vec{w}$ in $\mathbb{W}$

But again: $T(\underbrace{S(\vec{w})}_{=\vec{v}})=T(\vec{v})=\vec{w}$ by definition of S ✓

Special situation: $\dim V$ is finite & $\dim V = \dim W$

Then: $N(T) = \{0_V\}$ if and only if $R(T) = W$ by the Rank-Nullity Thm

Theorem 2: If $\dim V = \dim W = p$ then $T: V \to W$ linear map is invertible if & only if $N(T) = \{0_V\}$

Example: $\dim V = p$ $\quad W = \mathbb{R}^p$ $\quad T: V \longrightarrow \mathbb{R}^p$ is invertible
$\mathcal{B}$ basis for V $\quad \vec{v} \longmapsto [\vec{v}]_{\mathcal{B}}$

because $N(T) = \{0_V\}$.

(This was our $\boxed{\text{Main Example}}$ on page 5)

- Most notably, if $\dim V = \dim W = p$ we can always build an invertible linear transformation $T: V \longrightarrow W$

How? Pick $\mathcal{B}_1 = \{\vec{v}_1, \ldots, \vec{v}_p\}$ basis for V
$\mathcal{B}_2 = \{\vec{w}_1, \ldots, \vec{w}_p\}$ —— W

We define T by $\begin{cases} T(\vec{v}_1) = \vec{w}_1 \\ \vdots \\ T(\vec{v}_p) = \vec{w}_p \end{cases}$ since $\mathcal{B}_1$ is a basis, this assignment determines T uniquely ("Extend linearly using these assignments")

(How? $\boxed{T(\vec{v})} = T(\alpha_1 \vec{v}_1 + \cdots + \alpha_p \vec{v}_p) = \alpha_1 T(\vec{v}_1) + \cdots + \alpha_p T(\vec{v}_p)$
Write $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix}$ $\quad = \boxed{\alpha_1 \vec{w}_1 + \cdots + \alpha_p \vec{w}_p}$)

But by construction $R(T) = W$ so we automatically get

$\dim N(T) = \dim V - \dim R(T) = p - p = 0$ so $N(T) = \{0_V\}$

By Theorem 2, this map T is invertible.

[In symbols (Construction of T)

$$\begin{array}{ccc} V & \xrightarrow{F} & \mathbb{R}^p \\ T \downarrow & & \parallel \\ W & \xrightarrow{G} & \mathbb{R}^p \end{array}$$

$F(\vec{v}) = [v]_{\mathcal{B}_1}$ invertible

$G(\vec{w}) = [\vec{w}]_{\mathcal{B}_2}$ invertible

Our map is $T = G^{-1} \circ F : V \to \mathbb{R}^p \to W$. (Composition of invertible maps is invertible).