Lecture XXVII §5.9 Matrix representations for linear transformations

Recall: A linear map $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is always given by a matrix A of size $m \times n$. Indeed, $A = [T(e_1), \dots, T(e_n)]$ ($\{e_1, \dots e_n\}$ can. basis for $\mathbb{R}^n$)

Example: $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto \begin{bmatrix} x_1 - x_2 \\ 2x_1 + 4x_2 \\ x_2 \end{bmatrix} \quad \text{so} \quad A = \left[T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) \; T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) \right] = \begin{bmatrix} 1 & -1 \\ 2 & 4 \\ 0 & 1 \end{bmatrix}$$

TODAY'S GOAL: Find a similar way to interpret $T: \mathbb{V} \longrightarrow \mathbb{W}$ linear transformation where $\mathbb{V}$ & $\mathbb{W}$ are abstract vector spaces with $\dim \mathbb{V} = \mathbb{R}^n$, $\dim \mathbb{W} = \mathbb{R}^m$

§1 Three fundamental facts for $T: \mathbb{V} \longrightarrow \mathbb{W}$ linear

FACT 1 (Thm 1 Lecture 25, page 3) T is completely determined by its values on a basis for $\mathbb{V}$ (Same was true when $\mathbb{V} = \mathbb{R}^n$ & $\mathbb{W} = \mathbb{R}^m$)

FACT 2 (Main Example, Lecture 26, page 5):

Take $n = m$, $\mathbb{W} = \mathbb{R}^n$
Pick B basis for $\mathbb{V}$

$$T: \mathbb{V} \longrightarrow \mathbb{R}^n \qquad v \longmapsto [v]_B$$

"coordinates with respect to B"

This transformation is invertible & $T^{-1}: \mathbb{R}^n \longrightarrow \mathbb{V}$ where $B = \{\vec{v}_1, \dots, \vec{v}_n\}$
$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \longmapsto a_1\vec{v}_1 + \dots + a_n\vec{v}_n$$

Write $\boxed{T = T_B}$ to emphasis the fundamental role B plays here!

Upshot = By choosing a basis B for $\mathbb{V}$ we can always think of $\mathbb{V}$ as $\mathbb{R}^n$ (B allows us to have coordinates for each vector in $\mathbb{V}$, just as $\mathbb{R}^n$ has!)

Conclusion: If we choose coordinates for $\mathbb{V}$ & $\mathbb{W}$, by picking a basis for $\mathbb{V}$ & one for $\mathbb{W}$, then T can be though of as a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$.

FACT 3 (Table, Lecture 26, page 1) Composition of linear maps are linear

In particular:
$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "T_{B_{\mathbb{V}}}^{-1}", "\text{linear}"'] \arrow[rrr, bend right] & \mathbb{V} \arrow[r, "T", "\text{linear}"'] & \mathbb{W} \arrow[r, "T_{B_{\mathbb{W}}}", "\text{linear}"'] & \mathbb{R}^m \end{tikzcd}$$

$B_{\mathbb{V}}$ basis for $\mathbb{V}$
$B_{\mathbb{W}}$ basis for $\mathbb{W}$

So $\tilde{T}$ is linear. $\boxed{\tilde{T} = T_{B_{\mathbb{W}}} \circ T \circ T_{B_{\mathbb{V}}}^{-1}}$

<u>Note</u>: To emphasis that the choice of bases has on the construction, we write $\widetilde{T}$ as $T_{B_V B_W}$

Its $m \times n$ matrix will be denoted $[T]_{B_V, B_W}$

§2 Examples:

EXAMPLE 1 $\quad V = \mathcal{P}_3 \quad W = \mathcal{P}_2 \quad T: V \longrightarrow W$ is linear

$$P_{(x)} \longmapsto P'_{(x)}$$

Chose $B_{\mathcal{P}_3} = \{1, x, x^2, x^3\}$, $B_{\mathcal{P}_2} = \{1, x, x^2\}$

$$T_{B_V}: V \longrightarrow \mathbb{R}^4 \qquad T_{B_W}: W \longrightarrow \mathbb{R}^3$$

$$a+bx+cx^2+dx^3 \longmapsto \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \qquad \alpha+\beta x+\gamma x \longmapsto \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$T_{B_V}^{-1}: \mathbb{R}^4 \longrightarrow V$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \longmapsto a+bx+cx^2+dx^3$$

$$T_{B_{\mathcal{P}_2} B_{\mathcal{P}_3}}: \mathbb{R}^4 \xrightarrow{T_{B_{\mathcal{P}_3}}^{-1}} \mathcal{P}_3 \xrightarrow{T} \mathcal{P}_2 \xrightarrow{T_{B_{\mathcal{P}_2}}} \mathbb{R}^3$$

$$\boxed{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}} \longmapsto P = a+bx+cx^2+dx^3 \longmapsto P' = \underbrace{b}_{=\alpha} + \underbrace{\boxed{2c}}_{=\beta} x + \underbrace{\boxed{3d}}_{=\gamma} x^2 \longmapsto \boxed{\begin{bmatrix} b \\ 2c \\ 3d \end{bmatrix}}$$

<u>Conclude</u>: $T_{B_{\mathcal{P}_2} B_{\mathcal{P}_3}}\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} b \\ 2c \\ 3d \end{bmatrix}$ is a linear map $\mathbb{R}^4 \to \mathbb{R}^3$

Q: What's the matrix? Columns = values at e_1, e_2, e_3, e_4.

<u>A</u>: $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ 3×4 matrix

Q: Can we interpret these columns in terms of $B_{\mathcal{P}_2}$ & $B_{\mathcal{P}_3}$

<u>A</u>: YES!

Look at when T sends our basis $B_{\mathcal{P}_3}$

$$T(1) = 0 \rightsquigarrow [T(1)]_{B_{\mathcal{P}_2}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$T(x) = 1 \rightsquigarrow [T(x)]_{B_{\mathcal{P}_2}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T(x^2) = 2x \rightsquigarrow [T(x^2)]_{B_{\mathcal{P}_2}} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$T(x^3) = 3x^2 \rightsquigarrow [T(x^3)]_{B_{\mathcal{P}_2}} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

Obs: we get the columns of A!

Name: We call A the matrix for T relative to the bases B_{β_2} & B_{β_3}. Write $[T]_{B_{\beta_2} B_{\beta_3}}$

$\boxed{\text{EXAMPLE 2}}$ $V = M_{2\times 3}$, $W = \mathbb{R}^3$, $T: V \longrightarrow \mathbb{R}^3$

$$(a_{ij})_{ij} \longrightarrow \begin{bmatrix} 2a_{11} - 4a_{22} \\ a_{13} + a_{21} \\ a_{12} - a_{23} \end{bmatrix}$$

$$B_{M_{2\times3}} = \{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}\}$$

$$B_{\mathbb{R}^3} = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\} \rightsquigarrow T_{B_{\mathbb{R}^3}}: \mathbb{R}^3 \longrightarrow \mathbb{R}^3,\ \underline{x} \longmapsto \underline{x} \text{ is the identity!}$$

$$\tilde{T} = T_{B_{M_{2\times3}} B_{\mathbb{R}^3}} : \mathbb{R}^6 \xrightarrow{T_{B_{M_{2\times3}}}} M_{2\times 3} \xrightarrow{T} \mathbb{R}^3 \xrightarrow{T_{B_{\mathbb{R}^3}}} \mathbb{R}^3$$

$$\boxed{\begin{bmatrix} x_1\\x_2\\x_3\\x_4\\x_5\\x_6 \end{bmatrix}} \longmapsto \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{bmatrix} \longrightarrow \begin{bmatrix} 2x_1 - 4x_5 \\ x_3 + x_4 \\ x_2 - x_6 \end{bmatrix} \longrightarrow \boxed{\begin{bmatrix} 2x_1 - 4x_5 \\ x_3 + x_4 \\ x_2 - x_6 \end{bmatrix}}$$

Q: Matrix for $\tilde{T}$ $\mathbb{R}^6 \longrightarrow \mathbb{R}^3$, $\underline{x} \longmapsto \begin{bmatrix} 2x_1 - 4x_5 \\ x_3 + x_4 \\ x_2 - x_6 \end{bmatrix}$?

$$A = \underset{3\times 6}{\begin{bmatrix} 2 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}}$$

Write $[T]_{B_{M_{2\times3}} B_{\mathbb{R}^3}}$

Why?

$$\tilde{T}\left(\begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\0\\0\end{bmatrix},\ \tilde{T}\left(\begin{bmatrix}0\\1\\0\\0\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\\1\end{bmatrix},\ \tilde{T}\left(\begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix},\ \tilde{T}\left(\begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix},\ \tilde{T}\left(\begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-4\\0\\0\end{bmatrix},\ \tilde{T}\left(\begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\\-1\end{bmatrix}$$

$$\updownarrow$$

$$T\left(\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}\right) = \begin{bmatrix}2\\0\\0\end{bmatrix}\quad T\left(\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}\right)\quad T\left(\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix}\quad T\left(\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix}\quad T\left(\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}\right) = \begin{bmatrix}-4\\0\\0\end{bmatrix}\quad T\left(\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}\right) = \begin{bmatrix}0\\0\\-1\end{bmatrix}$$

Q: What if we choose a different basis for $\mathbb{R}^3$, for example

$$B'_{\mathbb{R}^3} = \left\{ \overset{w_1}{\begin{bmatrix}1\\0\\0\end{bmatrix}}, \overset{w_2}{\begin{bmatrix}1\\1\\0\end{bmatrix}}, \overset{w_3}{\begin{bmatrix}1\\1\\1\end{bmatrix}} \right\} ?$$

Then we need to write $[\ ]_{B'_{\mathbb{R}^3}}$ for the 6 columns in A.

New A (size 3×6) $= \begin{bmatrix} 2 & 0 & -1 & -1 & -4 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}$ Write $[T]_{B_{M_{2\times3}} B'_{\mathbb{R}^3}}$

because $\begin{bmatrix}2\\0\\0\end{bmatrix} = 2w_1$, $\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\1\\0\end{bmatrix} = w_3 - w_2$, $\begin{bmatrix}0\\0\\-1\end{bmatrix} = -w_3 + w_2$

$\begin{bmatrix}-4\\0\\0\end{bmatrix} = -4w_1$, $\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix} = w_2 - w_1$

§ 3. Matrix representation:

Now that we saw examples, we can state the main result of today's Lecture

Representation Theorem: Fix $T: \mathbb{V} \longrightarrow \mathbb{W}$ linear transformation with $\dim \mathbb{V} = n$ & $\dim \mathbb{W} = m$. Pick $B_{\mathbb{V}}$ = basis for $\mathbb{V}$ $= \{\vec{v}_1, \ldots, \vec{v}_n\}$

$B_{\mathbb{W}}$ = —— $\mathbb{W}$ $= \{\vec{w}_1, \ldots, \vec{w}_m\}$

Then, $T, B_{\mathbb{V}}$ & $B_{\mathbb{W}}$ give rise to a linear transformation

$$T_{B_{\mathbb{V}} B_{\mathbb{W}}} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

defined by $T_{B_{\mathbb{V}} B_{\mathbb{W}}}\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right) = T_{B_{\mathbb{W}}} \circ T \circ T^{-1}_{B_{\mathbb{V}}}\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right)$

Furthermore, its associated $m \times n$ matrix is obtain as follows:

$$\boxed{[T]_{B_{\mathbb{V}} B_{\mathbb{W}}}} = \Big[[T(v_1)]_{B_{\mathbb{W}}} \; \cdots \cdots \; [T(v_n)]_{B_{\mathbb{W}}} \Big]$$

Q: What does this matrix do for us?

A: Say we want to determine the vector $\vec{w} = T(\vec{v})$ for our favorite $\vec{v}$

Then: it suffices to compute the coordinates of $\vec{w}$ with respect to $B_{\mathbb{W}}$.

Formula:

$$\boxed{\underbrace{[T(\vec{v})]_{B_{\mathbb{W}}}}_{\text{size } m\times 1} = \underbrace{\boxed{[T]_{B_{\mathbb{V}} B_{\mathbb{W}}}}}_{\text{size } m \times n} \underbrace{[\vec{v}]_{B_{\mathbb{V}}}}_{\text{size } n \times 1}}$$

$[T]_{B_{\mathbb{V}} B_{\mathbb{W}}}$ matrix in the theorem!

$\rightsquigarrow$ If $\vec{v} = \alpha_1 \vec{v}_1 + \cdots + \alpha_n \vec{v}_n$ then $[v]_B = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$ & if $[T(\vec{v})]_{B_{\mathbb{W}}} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$, we get $\boxed{T(\vec{v}) = b_1 \vec{w}_1 + \cdots + b_m \vec{w}_m}$

[It is very important to keep the bases fixed!]

Back to our examples:

EXAMPLE 1 $T: \mathcal{P}_3 \longrightarrow \mathcal{P}_2$, $p(x) \longmapsto p'(x)$ $\quad B_1 = B_{\mathcal{P}_3} = \{1, x, x^2, x^3\}$ $\quad B_2 = B_{\mathcal{P}_2} = \{1, x, x^2\}$

$$[T]_{B_{\mathcal{P}_3} B_{\mathcal{P}_2}} = \Big[[T(1)]_{B_2} \; [T(x)]_{B_2} \; [T(x^2)]_{B_2} \; [T(x^3)]_{B_2} \Big] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

So $[T(7+5x-10x^3)]_{B_2} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \\ 0 \\ -10 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -30 \end{bmatrix} \rightsquigarrow T(7+5x-10x^3) = 5 - 30x^2$

EXAMPLE 2:

$$T: \mathcal{P}_2 \longrightarrow \mathbb{R}^2 \qquad a+bx+cx^2 \longmapsto \begin{bmatrix} c-b \\ 2c+a \end{bmatrix}$$

Find $[T]_{B_{\mathcal{P}_2} B_{\mathbb{R}^2}}$ when $B_{\mathcal{P}_2} = \{1, x, x^2\}$, $B_{\mathbb{R}^2} = \left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{w_1}, \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{w_2} \right\}$

$$T(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = w_1 + w_2 \rightsquigarrow [T(1)]_{\{w_1, w_2\}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$T(x) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = -\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -w_1 \rightsquigarrow [T(x)]_{\{w_1, w_2\}} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$T(x^2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2\begin{bmatrix} -1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 3w_1 + 2w_2 \rightsquigarrow [T(x^2)]_{\{w_1, w_2\}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Conclusion:
$$[T]_{B_{\mathcal{P}_2}, B_{\mathbb{R}^2}} = \left[[T(1)]_{\{w_1,w_2\}}, [T(x)]_{\{w_1,w_2\}}, [T(x^2)]_{\{w_1,w_2\}} \right] = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$$

$$\left[T(3+4x+5x^2) \right]_{\{w_1,w_2\}} = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix} \boxed{\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}} = \begin{bmatrix} 3-4+15 \\ 3+10 \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = [3+4x+5x^2]_{\{1,x,x^2\}}$$

So $T(3+4x+5x^2) = 14w_1 + 13w_2 = 14\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 13\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 13 \end{bmatrix}$

(This is what the formula defining T also tells us!)

EXAMPLE 3

$$T: M_{2\times 2} \longrightarrow \mathcal{P}_3 \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto (a+d)x^2 + ax + (b-3c)$$

Find $[T]_{B_{M_{2\times2}} B_{\mathcal{P}_3}}$ for $B_{M_{2\times2}} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$B_{\mathcal{P}_3} = \{1, x, x^2\}$

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) \underset{\substack{\downarrow \\ a=1\ b=c=d=0}}{=} 1x^2 + 0\cdot x + 0 = x^2 \rightsquigarrow [T(E_{11})]_{B_{\mathcal{P}_3}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \underset{\substack{\downarrow \\ a=0=c=d \\ b=1}}{=} 0\cdot x^2 + 0\cdot x + 1 = 1 \rightsquigarrow [T(E_{12})]_{B_{\mathcal{P}_3}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) \underset{\substack{\downarrow \\ a=b=d=0 \\ c=1}}{=} 0\cdot x^2 + 0\cdot x - 3 = -3 \rightsquigarrow [T(E_{21})]_{B_{\mathcal{P}_3}} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) \underset{\substack{\downarrow \\ a=b=c=0 \\ d=1}}{=} 1\cdot x^2 + 0\cdot x + 0 = x^2 \rightsquigarrow [T(E_{22})]_{B_{\mathcal{P}_3}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

So
$$[T]_{B_{M_{2\times2}} B_{\mathcal{P}_3}} = \begin{bmatrix} 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

§4 Algebraic Properties:

We know we have 3 operations for $T: \mathbb{V} \to \mathbb{W}$ (Table Lecture 26 page 1)

(I) Addition
(II) Scalar Multiplication
(III) Composition (whenever appropriate)

Q: What can we say about matrix representations for these 3 operations?

Fix $F: \mathbb{V} \longrightarrow \mathbb{W}$ linear transf
$G: \mathbb{V} \longrightarrow \mathbb{W}$

(I) $F+G: \mathbb{V} \longrightarrow \mathbb{W}$
$\vec{v} \longmapsto F(\vec{v}) + G(\vec{v})$
linear

(II) $\alpha F: \mathbb{V} \longrightarrow \mathbb{W}$
$\vec{v} \longmapsto \alpha F(\vec{v})$
linear

Fix bases $B_{\mathbb{V}}$ for $\mathbb{V}$ & $B_{\mathbb{W}}$ for $\mathbb{W}$

(I) Matrix Representation for Addition

Thm 1: $[F+G]_{B_{\mathbb{V}} B_{\mathbb{W}}} = [F]_{B_{\mathbb{V}} B_{\mathbb{W}}} + [G]_{B_{\mathbb{V}} B_{\mathbb{W}}}$

(Matrix of the sum is the sum of the matrices but we MUST use the same bases for the three matrices!)

(II) Matrix Representation for Scalar Multiplication

Thm 2: $[\alpha F]_{B_{\mathbb{V}} B_{\mathbb{W}}} = \alpha [F]_{B_{\mathbb{V}} B_{\mathbb{W}}}$

(Matrix of the scaled transformation is obtained by scaling the original matrix, but again we MUST use the same bases)

Examples: $F: \mathcal{P}_2 \longrightarrow \mathbb{R}$, $p \longmapsto p(1)$ $G: \mathcal{P}_2 \longrightarrow \mathbb{R}$, $p \longmapsto p'(2)$ $B_{\mathbb{V}} = \{1, x, x^2\}$, $B_{\mathbb{R}} = \{1\}$

$[F]_{B_{\mathbb{V}} B_{\mathbb{R}}} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$
$[G]_{B_{\mathbb{V}} B_{\mathbb{R}}} = \begin{bmatrix} 0 & 1 & 4 \end{bmatrix}$

$\leadsto$ $[F+G]_{B_{\mathbb{V}} B_{\mathbb{R}}} = \begin{bmatrix} 1 & 2 & 5 \end{bmatrix}$ & $[3F]_{B_{\mathbb{V}} B_{\mathbb{R}}} = \begin{bmatrix} 3 & 3 & 3 \end{bmatrix}$

(III) Matrix for compositions

$\dim \mathcal{V} = n \qquad \dim \mathcal{W} = p \qquad \dim \mathcal{U} = m$

$F : \mathcal{V} \longrightarrow \mathcal{W}$ linear

$G : \mathcal{W} \longrightarrow \mathcal{U}$ linear

$\rightsquigarrow \quad G \circ F : \mathcal{V} \xrightarrow{F} \mathcal{W} \xrightarrow{G} \mathcal{U}$

$\vec{v} \longmapsto F(\vec{v}) \to G(F(\vec{v}))$

is linear

Q: How to find a matrix representation for $G \circ F$?

- Fix $B_{\mathcal{V}}$ basis for $\mathcal{V}$ & $B_{\mathcal{U}}$ basis for $\mathcal{U}$.

$\rightsquigarrow$ What is $[G \circ F]_{B_{\mathcal{V}} B_{\mathcal{U}}}$?

A: We need to choose a basis $B_{\mathcal{W}}$ for $\mathcal{W}$ (think of it as a dummie variable)

$[F]_{B_{\mathcal{V}} B_{\mathcal{W}}}$ $\quad p \times n$ matrix

$[G]_{B_{\mathcal{W}} B_{\mathcal{U}}}$ $\quad m \times p$ matrix

Thm 3:

$$\underbrace{[G \circ F]_{B_{\mathcal{V}} B_{\mathcal{U}}}}_{m \times n} = \underbrace{[G]_{B_{\mathcal{W}} B_{\mathcal{U}}}}_{p \times m} \cdot \underbrace{[F]_{B_{\mathcal{V}} B_{\mathcal{W}}}}_{p \times n}$$

matrix multiplication

Why does this work?

$$[G \circ F(\vec{v})]_{B_{\mathcal{U}}} = [G(\boxed{F(\vec{v})})]_{B_{\mathcal{U}}} \overset{\text{Formula on page 4 applied to } G}{=} [G]_{B_{\mathcal{W}} B_{\mathcal{U}}} [F(\vec{v})]_{B_{\mathcal{W}}}$$

($F(\vec{v}) = \vec{w}$)

$$= [G]_{B_{\mathcal{W}} B_{\mathcal{U}}} \left([F]_{B_{\mathcal{V}} B_{\mathcal{W}}} [\vec{v}]_{B_{\mathcal{V}}} \right)$$

Formula on page 4 applied to F

$$\overset{\text{Assoc.}}{=} \left(\boxed{[G]_{B_{\mathcal{W}} B_{\mathcal{U}}} [F]_{B_{\mathcal{V}} B_{\mathcal{W}}}} \right) [\vec{v}]_{B_{\mathcal{V}}}$$

So the matrix $\boxed{[G]_{B_{\mathcal{W}} B_{\mathcal{U}}} [F]_{B_{\mathcal{V}} B_{\mathcal{W}}}}$ plays the same role as $[G \circ F]_{B_{\mathcal{V}} B_{\mathcal{U}}}$ did in the Formula on page 4, so these 2 matrices are the same!

Example: $F: M_{2\times 2} \longrightarrow \mathcal{P}_2$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto (a+d)x^2 + ax + (b-c)$ $\qquad G: \mathcal{P}_2 \longrightarrow \mathbb{R}^2$, $P(x) \longmapsto \begin{bmatrix} P(1) \\ P'(2) \end{bmatrix}$

$$(G\circ F)\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = G\Big(\underbrace{(a+d)\,x^2 + ax + (b-c)}_{=P}\Big) = \begin{bmatrix} (a+d)+a+(b-c) \\ 4(a+d)+a \end{bmatrix}$$

$P'_{(x)} = 2(a+d)x + a$

Choose $B_{M_{2\times 2}} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$B_{\mathbb{R}^2} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$\rightsquigarrow$ $(G\circ F)(E_{11}) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $(G\circ F)(E_{12}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$(G\circ F)(E_{21}) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $(G\circ F)(E_{22}) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$$[G\circ F]_{B_{M_{2\times 2}} B_{\mathbb{R}^2}} = \begin{bmatrix} 2 & 1 & -1 & 1 \\ 5 & 0 & 0 & 4 \end{bmatrix}.$$

Choose $B_{\mathcal{P}_2} = \{1, x, x^2\}$

$\rightsquigarrow$ $G(E_{11}) = x^2 + x \rightsquigarrow [G(E_{11})]_{B_{\mathcal{P}_2}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$G(E_{12}) = 1 \rightsquigarrow [G(E_{12})]_{B_{\mathcal{P}_2}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$G(E_{21}) = -1 \rightsquigarrow [G(E_{21})]_{B_{\mathcal{P}_2}} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

$G(E_{22}) = x^2 \rightsquigarrow [G(E_{22})]_{B_{\mathcal{P}_2}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

So $[G]_{B_{M_{2\times 2}} B_{\mathcal{P}_2}} = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

$\rightsquigarrow$ $F(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$F(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\rightsquigarrow$ $[F]_{B_{\mathcal{P}_2} B_{\mathbb{R}^2}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

$F(x^2) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

Check: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 & 1 \\ 5 & 0 & 0 & 4 \end{bmatrix}$ ✓

$[G]_{B_{\mathcal{P}_2} B_{\mathbb{R}^2}} \cdot [F]_{B_{M_{2\times 2}} B_{\mathcal{P}_2}} = [G\circ F]_{B_{M_{2\times 2}} B_{\mathbb{R}^2}}$

Application If $T: V \longrightarrow W$ is invertible, B_V basis for V, B_W — W

then $$[T^{-1}]_{B_W B_V} = \left([T]_{B_V B_W}\right)^{-1}$$

A: Invert the matrix & swap the role of B_V & B_W because $T^{-1}: W \longrightarrow V$