Lecture XXVIII: §6.2 Determinants

§1. What are determinants?

For each square matrix A we will define a number $\det(A)$.

- $\det(A)$ will be a polynomial in the entries of the matrix (the formulas will be discussed later & in future lectures)
- Main properties: (We'll discuss this later in the course)

① A is $\boxed{\text{singular}}$ (that it, non-invertible, or equivalently $N(A) \neq \{\vec{0}\}$) if and only if $\boxed{\det(A) = 0}$.

② det is multiplicative: $\det(AB) = \det(A)\det(B)$ for A, B of size $n \times n$

③ det is compatible with elementary row operations

④ A upper-triangular, that is $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & a_{n-1,n} \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}$, then

$\hookrightarrow$ zeros below the diagonal

$$\det(A) = a_{11} \cdots a_{nn}$$

In particular: $\det(\text{Identity matrix}) = 1$

⑤ $\det(A^T) = \det(A)$

§2. Formulas for 2x2:

Fix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ 2x2 matrix

Def The determinant of A is $\det A = a_{11}a_{22} - a_{12}a_{21}$

Ex: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \rightsquigarrow \det A = 1\cdot 4 - 2\cdot 3 = 4 - 6 = -2$.

$B = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix} \rightsquigarrow \det A = 3.8 - 4.6 = 24 - 24 = 0$

Notation: $\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ (use bars)

$+$ $-$

- take the product of the entries
- Sign $= \pm$ for the product

§3 Formulas for larger matrices:

To go from 2x2 to 3x3, 4x4, etc. we will employ an "inductive procedure". More precisely, we will define $\det(A)$ for an $n\times n$ matrix in terms of determinants of submatrices of A of size $(n-1)\times(n-1)$, by removing 1 row & 1 column of A

- Recall that we did this to define cross-products via the determinant of a 3x3 matrix.

$\boxed{\text{Definition = "Cofactors of } A\text{"}}$ Fix A of size $n\times n$ & two integers r, s $1\leq r\leq n$ (labelling a row)
$1\leq s\leq n$ (——— column)

- We let $\boxed{M_{r,s}}$ be the submatrix of A of size $(n-1)\times(n-1)$ obtained from A by removing row r & column s from A
- The number $\boxed{A_{r,s} = (-1)^{r+s}\det(M_{r,s})}$ is called the (r,s)-cofactor of A. Alternative name = signed minor
- The matrix $\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n}\\ \vdots & & & \\ A_{n1} & A_{n2} & \cdots & A_{nn}\end{pmatrix}$ of the cofactors of A

is called the cofactor matrix of A Name: $\mathrm{Cof}(A)$

Cofactor formula for det: $\boxed{\det(A) = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}}$

(multiply the 1st row of A by the 1st row of the cofactor matrix of A)

Idea: We will compute $\det A$ by "expanding along the 1st row of A"

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \underline{\underline{a_{11}}}\,(-1)^{\underline{1+1}} \underbrace{\begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & & \\ a_{n2} & \cdots & a_{nn} \end{vmatrix}}_{=A_{11}} + \underline{\underline{a_{12}}}\,(-1)^{\underline{1+2}} \underbrace{\begin{vmatrix} a_{21} & a_{23} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n3} & \cdots & a_{nn} \end{vmatrix}}_{=A_{12}}$$

(remove 1st row & 1st column from A) (remove 1st row & 2nd column from A)

$$+ \cdots + \underline{\underline{a_{1n}}}\,(-1)^{\underline{1+n}} \underbrace{\begin{vmatrix} a_{21} & \cdots & a_{2\,n-1} \\ \vdots & & \\ a_{n1} & \cdots & a_{n\,n-1} \end{vmatrix}}_{=A_{1n}}$$

(remove 1st row and last column from A)

Examples:

1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$\mathrm{Cof}(A)_{11} = \begin{bmatrix} \cancel{1} & \cancel{2} \\ \cancel{3} & \boxed{4} \end{bmatrix} = (-1)^{1+1}\, 4 = 4$

cross out the 1st row & 2nd column → left over submatrix

$\mathrm{Cof}(A)_{12} = \begin{bmatrix} \cancel{1} & \cancel{2} \\ 3 & \cancel{4} \end{bmatrix} = (-1)^{1+2}\, 3 = -3$

$\mathrm{Cof}(A)_{21} = \begin{bmatrix} \cancel{1} & 2 \\ \cancel{3} & \cancel{4} \end{bmatrix} = (-1)^{2+1}\, 2 = -2$

$\mathrm{Cof}(A)_{22} = \begin{bmatrix} 1 & \cancel{2} \\ \cancel{3} & \cancel{4} \end{bmatrix} = (-1)^{2+2}\, 1 = 1$

$\mathrm{Cof}(A) = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$

$$\det(A) = a_{11}\,\mathrm{Cof}(A)_{11} + a_{12}\,\mathrm{Cof}(A)_{12}$$
$$= 1 \cdot 4 + 2\,(-3) = 1 \cdot 4 - 2 \cdot 3 = -2$$

(as we had on page 1)

2. $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix} \rightsquigarrow \mathrm{Cof}(A) = ?$ has 9 entries

(we will use the formula for dets of 2x2 matrices from page 1)

<u>(1,1):</u> $\mathrm{Cof}(A)_{1,1} = (-1)^{1+1} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix} = +(1\cdot 1 - (-3)\cdot 0) = 1$

<u>(1,2):</u> $\mathrm{Cof}(A)_{1,2} = (-1)^{1+2} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix} = -(2\cdot 1 - (-3)\cdot 4) = -(2+12) = -14$

<u>(1,3):</u> $\mathrm{Cof}(A)_{1,3} = (-1)^{1+3} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix} = +(2\cdot 0 - 1\cdot 4) = -4$

<u>(2,1):</u> $\mathrm{Cof}(A)_{2,1} = (-1)^{2+1} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix} = -(2\cdot 1 - 1\cdot 0) = -2$

<u>(2,2):</u> $\mathrm{Cof}(A)_{2,2} = (-1)^{2+2} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix} = +(3\cdot 1 - 1\cdot 4) = 3-4 = -1$

<u>(2,3):</u> $\mathrm{Cof}(A)_{2,3} = (-1)^{2+3} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix} = -(3\cdot 0 - 2\cdot 4) = 8$

<u>(3,1):</u> $\mathrm{Cof}(A)_{3,1} = (-1)^{3+1} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix} = +(2\cdot(-3) - 1\cdot 1) = (-6-1) = -7$

<u>(3,2):</u> $\mathrm{Cof}(A)_{3,2} = (-1)^{3+2} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix} = -(3\cdot(-3) - 1\cdot 2) = -(-9-2) = 11$

$\underline{(3,3)}$: $\mathrm{Cof}(A)_{3,3} = (-1)^{3+3} \det \begin{bmatrix} 3 & 2 & \cancel{1} \\ 2 & 1 & \cancel{-3} \\ \cancel{4} & \cancel{0} & \cancel{1} \end{bmatrix} = +(3\cdot 1 - 2\cdot 2) = 3-4 = -1$

<u>Conclusion</u>: $\mathrm{Cof}(A) = \begin{bmatrix} 1 & -14 & -4 \\ -2 & -1 & 8 \\ -7 & 11 & -1 \end{bmatrix}$

$\det(A) = (\text{row } 1 \text{ of } A) \cdot (\text{row } 1 \text{ of } \mathrm{Cof}\, A) = [3,2,1][1,-14,-4] = 3 - 2\cdot 14 - 4$
$= -29$

<u>Note</u>: We don't need to compute the full cofactor matrix if we only want to know $\det(A)$. Knowing the 1^{st} row of $\mathrm{Cof}(A)$ is all we need!

<u>Q</u>: Why do we care about $\mathrm{Cof}(A)$?

<u>A</u>: We have

$$(*) \quad \boxed{A \cdot \mathrm{Cof}(A)^T = \mathrm{Cof}(A)^T A = \det(A) \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}}$$

In particular, when A is invertible, then $A^{-1} = \dfrac{\mathrm{Cof}(A)^T}{\det(A)}$ Name: $\mathrm{Cof}(A)^T$ = Adjoint matrix

We saw this formula worked for size 2x2 in Lecture 8 since: $\begin{cases} \mathrm{Cof}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} -d & c \\ b & -a \end{pmatrix}^T = \begin{pmatrix} -d & b \\ c & a \end{pmatrix} \\ \Delta = ad - bc = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{cases}$

- Let's check the identity $(*)$ in our examples:

① $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\mathrm{Cof}(A) = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$ $\rightsquigarrow$ $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$

$\det A = -2$ $= (-2) I_2$

② $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix}$ $\mathrm{Cof}(A)^T = \begin{bmatrix} 1 & -2 & -7 \\ -14 & -1 & 11 \\ -4 & 8 & -1 \end{bmatrix}$ $\rightsquigarrow$ $\begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -7 \\ -14 & -1 & 11 \\ -4 & 8 & -1 \end{bmatrix} = \begin{bmatrix} -29 & 0 & 0 \\ 0 & -29 & 0 \\ 0 & 0 & -29 \end{bmatrix}$

$\det A = -29$ $= -29\, I_3$

<u>One last example</u>: We use the cofactor formula to compute the determinant of a 4x4 matrix:

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ -1 & 2 & 3 & 1 \\ -3 & 2 & -1 & 0 \\ 2 & -3 & -2 & 1 \end{bmatrix}$$

We need to compute $Cof(A)_{11}$, $Cof(A)_{12}$ & $Cof(A)_{14}$ ($a_{13}=0$ so we don't need $Cof(A)_{13}$)

$\rightsquigarrow \det A = 1\, Cof A_{1,1} + 2\, Cof A_{1,2} + 2\, Cof A_{1,4} = (-15) + 2(-18) + 2(-6)$
$= -15 - 36 - 12 = \boxed{-63}$

• $$Cof(A)_{1,1} = (-1)^{1+1} \begin{vmatrix} \cancel{1} & \cancel{2} & \cancel{0} & \cancel{2} \\ \cancel{-1} & 2 & 3 & 1 \\ \cancel{-3} & 2 & -1 & 0 \\ \cancel{2} & -3 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 1 \\ 2 & -1 & 0 \\ -3 & -2 & 1 \end{vmatrix} = B_1 \quad \rightsquigarrow = \boxed{-15}$$

scalar is a_{1j}

$$\det B_1 = (-1)^{1+1}\, 2 \begin{vmatrix} 2 & 3 & 1 \\ 2 & -1 & 0 \\ -3 & -2 & 1 \end{vmatrix} + (-1)^{1+2}\, 3 \begin{vmatrix} 2 & 3 & 1 \\ 2 & -1 & 0 \\ -3 & -2 & 1 \end{vmatrix} + (-1)^{1+3}\, 1 \begin{vmatrix} 2 & 3 & 1 \\ 2 & -1 & 0 \\ -3 & -2 & 1 \end{vmatrix}$$

$$= 2\big((-1)\cdot 1 - 0\cdot(-2)\big) - 3\big(2\cdot 1 - 0\cdot(-3)\big) + 1\big(2\cdot(-2) - (-1)(-3)\big)$$

$$= 2(-1) - 3(2) + (-4-3) = -2 - 6 - 7 = \boxed{-15}$$

• $$Cof(A)_{1,2} = (-1)^{1+2} \begin{vmatrix} \cancel{1} & \cancel{2} & \cancel{0} & \cancel{2} \\ -1 & \cancel{2} & 3 & 1 \\ -3 & \cancel{2} & -1 & 0 \\ 2 & \cancel{-3} & -2 & 1 \end{vmatrix} = (-1) \underbrace{\begin{vmatrix} -1 & 3 & 1 \\ -3 & -1 & 0 \\ 2 & -2 & 1 \end{vmatrix}}_{=B_2} \quad \rightsquigarrow = -(18) = \boxed{-18}$$

$$\det(B_2) = (-1)^{1+1}(-1) \begin{vmatrix} -1 & 3 & 1 \\ -3 & -1 & 0 \\ 2 & -2 & 1 \end{vmatrix} + (-1)^{1+2}\, 3 \cdot \begin{vmatrix} -1 & 3 & 1 \\ -3 & -1 & 0 \\ 2 & -2 & 1 \end{vmatrix} + (-1)^{1+3}\, 1 \cdot \begin{vmatrix} -1 & 3 & 1 \\ -3 & -1 & 0 \\ 2 & -2 & 1 \end{vmatrix}$$

$$= (-1)\big((-1)\cdot 1 - 0\cdot(-2)\big) - 3\big((-3)\cdot 1 - 0\cdot 2\big) + 1\big((-3)(-2) - (-1)\cdot 2\big)$$

$$= (-1)(-1) - 3(-3) + 1(6+2) = 1 + 9 + 8 = \boxed{18}$$

• $$Cof(A)_{1,4} = (-1)^{1+4} \begin{vmatrix} \cancel{1} & \cancel{2} & \cancel{0} & \cancel{2} \\ -1 & 2 & 3 & \cancel{1} \\ -3 & 2 & -1 & \cancel{0} \\ 2 & -3 & -2 & \cancel{1} \end{vmatrix} = - \underbrace{\begin{vmatrix} -1 & 2 & 3 \\ -3 & 2 & -1 \\ 2 & -3 & -2 \end{vmatrix}}_{=B_3} \quad \rightsquigarrow = \boxed{-6}$$

$$\det B_3 = (-1)^{1+1}(-1) \begin{vmatrix} -1 & 2 & 3 \\ -3 & 2 & -1 \\ 2 & -3 & -2 \end{vmatrix} + (-1)^{1+2}\, 2 \begin{vmatrix} -1 & 2 & 3 \\ -3 & 2 & -1 \\ 2 & -3 & -2 \end{vmatrix} + (-1)^{1+3}\, 3 \begin{vmatrix} -1 & 2 & 3 \\ -3 & 2 & -1 \\ 2 & -3 & -2 \end{vmatrix}$$

$$= +(-1)\big(2\cdot(-2) - (-1)(-3)\big) - 2\big((-3)(-2) - (-1)\cdot 2\big) + 3\big((-3)(-3) - 2\cdot 2\big)$$

$$= (-1)(-4-3) - 2(6+2) + 3(9-4) = 7 - 16 + 15 = \boxed{6}$$

Special case: Triangular matrices

Def A of size $n \times n$ is <u>upper triangular</u> if $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & a_{n-1,n} \\ 0 & \cdots & 0 & a_{n,n} \end{bmatrix}$

↳ zeros <u>below</u> the diagonal

Def A of size $n \times n$ is <u>lower triangular</u> if $A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ a_{n,1} & \cdots & a_{n,n-1} & a_{nn} \end{bmatrix}$

↳ zeros <u>above</u> the diagonal

Def A is <u>triangular</u> if it is upper or lower triangular

Theorem: If A is triangular, then $\det A = a_{11} a_{22} \cdots a_{nn}$
$=$ product of diagonal entries of A

Proof: Show it for <u>lower triangular</u>

$$\det \begin{pmatrix} \boxed{a_{11}} & 0 & \cdots & 0 \\ a_{12} & \ddots & & \\ \vdots & & \ddots & \\ a_{1n} & \cdots & & a_{nn} \end{pmatrix} = a_{1,1} (-1)^{1+1} \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{12} & a_{22} & 0 \cdots & 0 \\ \vdots & & \ddots & 0 \\ a_{1n} & \cdots & & a_{nn} \end{vmatrix}$$

$$= a_{1,1} \begin{vmatrix} a_{22} & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ a_{n2} & \cdots & & a_{nn} \end{vmatrix}$$

↳ this matrix has size 1 less & it's also lower triangular, so we can repeat the process

$$= a_{1,1} \left(+ a_{22} \begin{vmatrix} a_{22} & 0 & \cdots & & 0 \\ \vdots & a_{33} & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ \vdots & & & & 0 \\ a_{n2} & \cdots & & & a_{nn} \end{vmatrix} \right) = a_{1,1} a_{2,2} \begin{vmatrix} a_{33} & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ \vdots & & & 0 \\ a_{n3} & \cdots & & a_{nn} \end{vmatrix}$$

Continuing in this way we reach $a_{1,1} a_{2,2} a_{3,3} \cdots a_{n,n}$.