Lecture XXIX: §6.3 Elementary Operations & Determinants

Recall: Given an $n\times n$ matrix A, we compute $\det(A)$ = real number using the cofactor formula:

$$\det(A) = a_{11}(-1)^{1+1}\det M_{11} + a_{12}(-1)^{1+2}\det M_{12} + \cdots + a_{1n}(-1)^{1+n}\det M_{1n}$$

where $M_{r,s}$ = matrix of size $(n-1)\times(n-1)$ obtains by removing row r & column s from A

Example: $\det\begin{pmatrix}1&0&2\\1&1&2\\3&4&5\end{pmatrix} = 1\cdot(-1)^{1+1}\det\begin{pmatrix}1&2\\4&5\end{pmatrix} + 0(-1)^{1+2}\det\begin{vmatrix}1&2\\3&5\end{vmatrix} + 2(-1)^{1+3}\det\begin{pmatrix}1&1\\3&4\end{pmatrix}$

$= (1\cdot 5 - 2\cdot 4) + 0 + 2(1\cdot 4 - 1\cdot 3) = (5-8) + 2(4-3)$

$= -3 + 2 = \boxed{-1}$

- When A is triangular $\left(\begin{pmatrix}a_{11}&&*\\&\ddots&\\0&&a_{nn}\end{pmatrix} \text{ or } \begin{pmatrix}a_{11}&&0\\&\ddots&\\ *&&a_{nn}\end{pmatrix}\right)$ then $\det A = a_{11}a_{22}\cdots a_{nn}$

TODAY: Use elementary row operations to simplify the calculation of determinants

3 row operations $\longleftrightarrow$ 3 effects on determinants

We will go through each case separately:

$\left(A \xrightarrow[\text{operation}]{} B\right)$

Main Theorem: Call A the input & B the output matrix, so $A \xrightarrow[\text{I, II or III}]{\text{operation}} B$

Then, we can relate $\det(A)$ to $\det(B)$ as follows:

Operation	$\det(B) =$	Effect
(I) $R_i \longleftrightarrow R_j$, $i\neq j$	$-\det(A)$	switch signs
(II) $R_i \to \alpha R_i$, $\alpha \neq 0$ in $\mathbb{R}$	$\alpha \det(A)$	multiply by scalar α
(III) $R_i \to R_i + aR_j$, $i\neq j$, a real number	$\det(A)$	no change

Next, we discuss each case & do examples. We use the same input matrix A in all examples.

§1 Operation (I) Exchange 2 rows

Effect: $\det(B) = -\det(A)$

Example: $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{} B = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 2 \\ 3 & 4 & 5 \end{pmatrix}$

$$\det(A) = +\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} = (5-8) + 2(4-3) = -3+2 = -1$$

$$\det(B) = +\begin{vmatrix} 0 & 2 \\ 4 & 5 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} + 2\begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} = -8 - (5-6) + 2\cdot 4 = -8 + 1 + 8 = \boxed{1}$$

Check: $\det(B) = +1 = -\det(A)$ (as was predicted by the Theorem)

§2. Operation II: Multiply a row by a nonzero scalar α

Effect: $\det(B) = \alpha \det(A)$

Example: $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix} \xrightarrow[R_2 \to 3R_2]{} B = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 3 & 6 \\ 3 & 4 & 5 \end{pmatrix}$

$$\det(B) = 1\begin{vmatrix} 3 & 6 \\ 4 & 5 \end{vmatrix} - 0\begin{vmatrix} 3 & 6 \\ 3 & 5 \end{vmatrix} + 2\begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix} = (15-24) - 0 + 2(12-9) = -9 + 6 = -3$$

Check: $\det(B) = -3 = 3(-1) = 3\det(A)$ (as predicted by the Theorem)

Note $\begin{vmatrix} 3 & 6 \\ 4 & 5 \end{vmatrix} = 3 \det\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$ & $\begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix} = 3 \det\begin{vmatrix} 1 & 1 \\ 4 & 5 \end{vmatrix}$

This explains why the scalar α appears in the formula for $\det B$:

(1) If α multiplies the 1st row, then α will appear in $b_{1i} = \alpha a_{1i}$

(2) ——— another row, ——— in the smaller determinants

Consequence: If we scale the whole matrix by a number c, then:

$$\det(cA) = c^{n} \det(A) \quad \text{if } A \text{ has size } n \times n$$

[Why? Because we scale each row by c, so we get n factors c, one per row]

§3. Operation (III): Replace now R_i by $R_i + \alpha R_j$ for $j \neq i$, α in $\mathbb{R}$

Effect: $\det(B) = \det(A)$

Example: $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix} \xrightarrow[R_2 \to R_2 - R_1]{} B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 3 & 4 & 5 \end{pmatrix}$

$$\det B = +1 \begin{vmatrix} 1 & 0 \\ 4 & 5 \end{vmatrix} - 0 \begin{vmatrix} 0 & 0 \\ 3 & 5 \end{vmatrix} + 2 \begin{vmatrix} 0 & 1 \\ 3 & 4 \end{vmatrix} = 5 - 0 + 2(-3) = -1$$

Check: $\det B = -1 = \det(A)$

§4 Combine all operations

We can use the elementary operations to find $A \sim A'$ with A' in EF

Since an EF matrix is upper triangular, its determinant will be very easy to compute (it's just the product of its diagonal entries)

To recover $\det(A)$ from $\det(A')$ we need to keep track of which operations we use & remember the effect on the determinant

⇝ Algorithm for computing $\det(A)$

(1) Find $A \sim A'$ A' in EF

(2) Compute $\det(A')$

(3) Use the recorded operations to recover $\det(A)$ from $\det(A')$

We do one example:

Example: $A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 4 & 8 & 4 \\ -3 & 2 & -1 & 0 \\ 2 & -3 & -2 & 1 \end{bmatrix}$ Compute $\det(A)$

- At each step $A_i \longrightarrow A_{i+1}$ of the row reduction, we keep track of the changes in the determinant. (ie write the new determinant in terms of $\det(A)$.)

- At the very end, we will get $\beta \det(A) = \det(A')$ & since we can easily compute $\det(A')$ using the formula for triangular matrices & we have $\beta \neq 0$, we'll be able to solve for $\det A$.

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 4 & 8 & 4 \\ -3 & 2 & -1 & 0 \\ 2 & -3 & -2 & 1 \end{bmatrix} \xrightarrow[\text{Operation III}]{R_3 \to R_3 + 3R_1} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 4 & 8 & 4 \\ 0 & 8 & -1 & 6 \\ 2 & -3 & -2 & 1 \end{bmatrix} \xrightarrow[\text{Operation III}]{R_4 \to R_4 - 2R_1} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 4 & 8 & 4 \\ 0 & 8 & -1 & 6 \\ 0 & -7 & -2 & -3 \end{bmatrix}$$

$\det(A)$ $\quad$ $\det(A)$ $\quad$ $\det(A)$

$$\xrightarrow[\text{Operation III}]{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 4 & 8 & 4 \\ 0 & 0 & -17 & -2 \\ 0 & -7 & -2 & -3 \end{bmatrix} \xrightarrow[\text{Operation (II)}]{R_2 \to R_2/4} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -17 & -2 \\ 0 & -7 & -2 & -3 \end{bmatrix} \xrightarrow[\text{Operation III}]{R_4 \to R_4 + 7R_2} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -17 & -2 \\ 0 & 0 & 12 & 4 \end{bmatrix}$$

$\det(A)$ $\quad$ $\frac{1}{4}\det(A)$ $\quad$ $\frac{1}{4}\det(A)$

$$\xrightarrow[\text{Operation (II)}]{R_3 \to \frac{-1}{17}R_3} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & +2/17 \\ 0 & 0 & 12 & 4 \end{bmatrix} \xrightarrow[\text{Operation III}]{R_4 \to R_4 - 12R_3} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2/17 \\ 0 & 0 & 0 & 44/17 \end{bmatrix} = A' \quad \text{triangular!}$$

$\frac{-1}{17}\left(\frac{1}{4}\det A\right)$ $\quad$ $\frac{-1}{68}\det(A)$

$$\det A' = 1 \cdot 1 \cdot 1 \, \frac{44}{17} = \frac{44}{17} = \frac{-1}{68}\det(A)$$

Conclude: $\det(A) = \frac{-44 \cdot 68}{17} = \boxed{-176}$

Another example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - 7R_1 \\ \text{Operation III}}]{R_2 \to R_2 - 4R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -11 \end{bmatrix} \xrightarrow[\text{Operation III}]{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{Op. II}]{R_2 \to \frac{R_2}{-3}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = A'$$

$\det(A)$ $\quad$ $\det A$ $\quad$ $\det A$ $\quad$ $\frac{-1}{3}\det A$

$\det A' = 1 \cdot 1 \cdot 1 = \frac{-1}{3}\det A$, so $\boxed{\det A = -3}$

§5 Determinants of Non-singular matrices:

Theorem: A of size $n\times n$ is invertible if, and only if $\det A \neq 0$.

Why is this?

Recall: Algorithm to produce A^{-1} or show A^{-1} does not exist

$$(*) \quad \left(A \;\middle|\; I_n \right) \sim \left(A' \;\middle|\; B \right) \qquad A' \text{ REF}$$

- If A is invertible, we get $A' = I_n$
- If A is NOT invertible, we get A' has a row of zeroes at the end

In particular, the same operations we used in $(*)$ give $A \sim A'$

No matter what the exact operations we use, we always get

$\boxed{\det A' = \beta \det(A)}$ for some $\boxed{\text{nonzero } \beta}$ (in our examples from before, we had A' EF & $\beta = \frac{-1}{68}$ & $\frac{-1}{3}$)

- If A is invertible, then $A' = I_n$ so $\det A' = \det\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = (1)^n = 1$

  Conclude: $\det(A) = \dfrac{\det A'}{\beta} = \dfrac{1}{\beta} \neq 0$

- If A is not invertible then $A' = \begin{pmatrix} 1 & & & * \\ & 1 & & \\ & & \ddots & \\ 0 & & & \boxed{0} \end{pmatrix}$

  A' is Triangular & has a 0 in the diagonal

  so $\det A' = 1 \cdot 1 \cdot \dots \cdot 0$

  Conclude: $\det(A) = \dfrac{\det A'}{\beta} = \dfrac{0}{\beta} = 0$

We showed that invertible matrices have determinant $\neq 0$ & all matrices with $\det = 0$ are non-invertible.

This is exactly what we wanted to show! ☺