Lecture XXX : §6.4 Cramer's Rule

Recall: Row operations change determinants in specific ways:

(I) $A \xrightarrow[R_i \leftrightarrow R_j \ (i\neq j)]{} B$ $\quad \det(B) = -\det(A)$

(II) $A \xrightarrow[R_i \to cR_i \ (c\neq 0)]{} B$ $\quad \det(B) = c\det(A)$

(III) $A \xrightarrow[R_i \to R_i + cR_j \ \ i\neq j]{} B$ $\quad \det(B) = \det(A)$

These properties allowed us to derive an Algorithm to compute $\det(A)$ via $A \sim A'$ in EF (so, triangular!).

The following result will allow us to get similar rules for "column operations"

- **Theorem 1:** $\det(A^T) = \det(A)$

§1. Product Rule:

One of the fundamental properties of $\det(\ )$ is that it plays nicely with products.

Theorem 2: A, B of size $n\times n$, then $\det(AB) = \det(A)\det(B)$

Consequence: If A is invertible, then $\det(A^{-1})\det(A) = \det(I_n) = 1$

so both $\det(A)\neq 0$, $\det(A^{-1})\neq 0$ & $\det(A^{-1}) = \dfrac{1}{\det A}$.

In particular, if $\det(A)=0$, A cannot be invertible

(We saw a proof of this last time!)

Example: $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ $\quad \det(A) = 2\cdot 3 - 2 = 4 \neq 0$

$B = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$ $\quad \det(B) = 2 - 3 = -1 \neq 0$

$\det A \det B = \boxed{-4}$ as expected!!

$$AB = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -2+2 & 2\cdot 3-4 \\ -1+3 & 3-6 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & -3 \end{bmatrix} \rightsquigarrow \det = \boxed{-4}$$

Q: How to prove the product rule? We'll need the following lemma

Lemma: If A is singular of size $n \times n$ & B is any matrix of size $n \times n$, then AB is also singular.

$\left(\text{so } \det(AB) = 0 = \underbrace{\det A}_{=0} \det B \right)$

Proof: We will use that if C has size $n \times n$ then C is singular if, and only if, C^T is also singular

(This is because for invertible matrices $(C^{-1})^T = (C^T)^{-1}$ & we know invertible is the same as non-singular)

- Since A is singular, A^T is also singular
- Then $B^T A^T$ is singular (Pick any $\vec{v}$ in $\mathbb{R}^n$ $\vec{v} \neq \vec{0}$ with $A^T \cdot \vec{v} = \vec{0}$, then $(B^T A^T) v = B^T (A^T v) = B^T \vec{0} = \vec{0}$ so $B^T A^T$ is singular).
- Now, take $C = B^T A^T$. It is singular, so C^T is also singular. But $C^T = (B^T A^T)^T = AB$

Conclusion: If A is singular, then AB is also singular. $\square$

Proof of Product Rule:

- If A is singular, the Lemma says AB is also singular. Since the determinant of singular matrices is 0, we get $\det(AB) = 0 = 0 \cdot \det B = \det A \cdot \det B$, so the

Product Rule works if A is singular.

- Next, we assume A is nonsingular, so $A \sim I_n$ & By our rules of determinant under row operations, we get $\beta \det(A) = \det(I_n) = 1$ so $\det(A) = \frac{1}{\beta}$

$\neq 0$

(*) But, the same row operations giving $A \sim I_n$ produce $AB \sim B$ so $\beta \det(AB) = \det B$ with $\beta \neq 0$

Then $\beta \det(AB) = \det B$

$$\det(AB) = \frac{\det B}{\beta} = \frac{1}{\beta} \det B = \det A \det B$$

Conclusion: $\boxed{\det(AB) = \det A \det B}$ $\square$

- Let's check our claim (*) about the row operations on an example

$\boxed{\text{EXAMPLE}}$ (from page 1) $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ & $B = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$ $\leadsto AB = \begin{bmatrix} 0 & 2 \\ 2 & -3 \end{bmatrix}$

$$\underset{\det A}{\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}} \xrightarrow[R_1 \leftrightarrow R_2]{} \underset{-\det A}{\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}} \xrightarrow[R_2 \to R_2 - 2R_1]{} \underset{-\det A}{\begin{bmatrix} 1 & 3 \\ 0 & -4 \end{bmatrix}} \xrightarrow[R_2 \to \frac{R_2}{-4}]{} \underset{\frac{-1}{4}(-\det A)}{\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}} \xrightarrow[R_1 \to R_1 - 3R_2]{} \underset{\frac{\det A}{4}}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = I_2$$

We apply the same operations to AB & see that we get B at the end

$$AB = \begin{bmatrix} 0 & 2 \\ 2 & -3 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{} \begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix} \xrightarrow[R_2 \to R_2 - 2R_1]{} \begin{bmatrix} 2 & -3 \\ -4 & 8 \end{bmatrix} \xrightarrow[R_2 \to \frac{R_2}{-4}]{} \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \xrightarrow[R_1 \to R_1 - 3R_2]{} \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} = B$$

So the claim works in the example!

§2. Cramer's Rule:

GOAL: Use determinants to find the unique solution to $A\underline{x} = \underline{b}$ where A is a nonsingular $n\times n$ matrix.

- The formula is provided by Cramer's Rule:

Theorem 3 (Cramer's Rule): Fix $A = [A_1 \cdots A_n]$ a nonsingular $n\times n$ matrix and a vector $\underline{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ in $\mathbb{R}^n$.

- For each $i = 1, \ldots, n$, we build a new matrix
$$B_i = [A_1 \cdots A_{i-1}\ b\ A_{i+1} \cdots A_n]$$
(we replace the i^{th} column of the A matrix with the column vector $\underline{b}$)

- Then, the unique solution to $A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ satisfies:
$$x_1 = \frac{\det(B_1)}{\det(A)}, \quad x_2 = \frac{\det(B_2)}{\det(A)}, \ldots, \quad x_n = \frac{\det(B_n)}{\det(A)}$$

Example ① Solve $\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\begin{array}{cc} \underline{b} & \\ \end{array}$$
$$B_1 = \begin{bmatrix} \boxed{\begin{matrix} 1 \\ 2 \end{matrix}} & \begin{matrix} 2 \\ 3 \end{matrix} \end{bmatrix} \rightsquigarrow \det B_1 = 3-4 = -1$$

$$B_2 = \begin{bmatrix} \begin{matrix} 2 \\ 1 \end{matrix} & \boxed{\begin{matrix} 1 \\ 2 \end{matrix}} \end{bmatrix} \rightsquigarrow \det B_2 = 4-1 = 3$$

$$\det(A) = 6-2 = 4$$

So Solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ satisfies $x_1 = \frac{\det B_1}{\det A} = \frac{-1}{4}$

$x_2 = \frac{\det B_2}{\det A} = \frac{3}{4}$

Check: $\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -1/4 \\ 3/4 \end{bmatrix} = \begin{bmatrix} \frac{-2+6}{4} \\ \frac{-1+9}{4} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ✓

Example ② Solve $\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$B_1 = \begin{bmatrix} \boxed{\begin{matrix} 3 \\ 4 \end{matrix}} & \begin{matrix} 2 \\ 3 \end{matrix} \end{bmatrix} \rightsquigarrow \det B_1 = 9 - 8 = 1 \rightsquigarrow x_1 = \frac{1}{4}$

$B_2 = \begin{bmatrix} \begin{matrix} 2 \\ 1 \end{matrix} & \boxed{\begin{matrix} 3 \\ 4 \end{matrix}} \end{bmatrix} \rightsquigarrow \det B_2 = 8 - 3 = 5 \rightsquigarrow x_2 = \frac{5}{4}$

$\det A = 4$

Check: $\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1/4 \\ 5/4 \end{bmatrix} = \begin{bmatrix} \frac{2+10}{4} \\ \frac{1+15}{4} \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ ✓

Proof of Cramer's Rule: (Optional)

We prove the formula for x_1 is correct. The others follow by the same reasoning.

For starters, we know we have a solution so

$$\begin{array}{ccl} A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} & = & x_1 \operatorname{col}_1 A + x_2 \operatorname{col}_2(A) + \cdots + x_n \operatorname{col}_n(A) \\ \| & & \\ b & = & x_1 A_1 + x_2 A_2 + \cdots + x_n A_n \end{array}$$

Next, we transpose this expression:

$$b^T = (x_1 A_1 + x_2 A_2 + \cdots + x_n A_n)^T$$

scalars!

$$\boxed{b^T = x_1 A_1^T + x_2 A_2^T + \cdots + x_n A_n^T} \quad (*)$$

$$[b_1 \cdots b_n] = [x_1 \cdots x_n] \begin{bmatrix} A_1^T \\ \vdots \\ A_n^T \end{bmatrix} = [x_1 \cdots x_n] \, A^T$$

We build a new matrix C from A^T, by multiplying its 1^{st} row by x_1:

$$C = \begin{bmatrix} x_1 A_1^T \\ A_2^T \\ \vdots \\ A_n^T \end{bmatrix}$$

Now, we compute det(C) using Row Operation II for scalar $= x_1$

$$\boxed{\det(C)} = x_1 \det \begin{pmatrix} A_1^T \\ A_2^T \\ \vdots \\ A_n^T \end{pmatrix} = x_1 \det(A^T) \underset{\substack{\uparrow \\ \text{Thm 1}}}{=} \boxed{x_1 \det(A)} \quad (*)$$

Using (*) we see that

$$x_1 A_1^T = b^T - x_2 A_2^T - \cdots - x_n A_n^T$$

So
$$\det(C) = \det \begin{pmatrix} b^T - x_2 A_2^T - \cdots - x_n A_n^T \\ A_2^T \\ \vdots \\ A_n^T \end{pmatrix}$$

Next, we perform the row operation

$$R_1 \longrightarrow R_1 + x_2 R_2 + x_3 R_3 + \cdots + x_n R_n$$

that doesn't change the determinant.

The first row of the new matrix becomes b^T

<u>Conclusion</u>: $\det(C) = \det \begin{pmatrix} b^T \\ A_2^T \\ \vdots \\ A_n^T \end{pmatrix} \underset{\substack{\uparrow \\ \text{Thm 1}}}{=} \det \begin{pmatrix} b^T \\ A_2^T \\ \vdots \\ A_n^T \end{pmatrix}^T$

$$= \det \big(\underbrace{b \; A_2 \cdots A_n}_{\text{matrix } B_1} \big)$$

So $\det B_1 = \det C = x_1 \underbrace{\det A}_{\neq 0}$ by (*)

So $\boxed{x_1 = \dfrac{\det B_1}{\det A}}$ as we wanted to show.