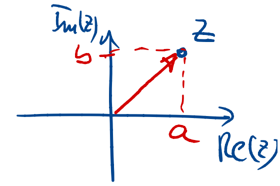
Lecture XXXV: §4.6 Complex eigenvectors & eigenspaces
Real symmetric matrices



Recall (Complex numbers) $z = a + ib$ $\quad a = \mathrm{Re}(z)$ in $\mathbb{R}$, $b = \mathrm{Im}(z)$ in $\mathbb{R}$

- i satisfies $i^2 = -1$
- Addition: $(a+ib) + (c+id) = (a+c) + i(b+d)$
- Multiplication: $(a+ib)(c+id) = (ac-bd) + i(ad+bc)$
- Complex conjugation $\bar{z} = a - ib$ $\quad (\overline{zw} = \bar{z}\,\bar{w}, \ \overline{z+w} = \bar{z} + \bar{w}$
- Norm (or modulus) $|z| = \sqrt{a^2+b^2}$ satisfies $|z| = z \cdot \bar{z}$, $\ |zw| = |z|\,|w|$

So for $z \neq 0$ $\quad \boxed{z^{-1} = \dfrac{\bar{z}}{|z|^2}}$

Fundamental Theorem: Every polynomial in $\mathbb{C}[x]$ of degree ≥ 1 has a root in $\mathbb{C}$

Consequence: Roots of polynomials in $\mathbb{R}[x]$ = real or conjugate pairs (z & $\bar{z}$).

§1 Vectors in $\mathbb{C}^n$:

Vectors will be define just as we did for $\mathbb{R}^n$, except now entries are in $\mathbb{C}$, not just in $\mathbb{R}$.

We write $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ & impose $v_1, \ldots, v_n$ are in $\mathbb{C}$.

Note: By construction, $\mathbb{R}^n \subseteq \mathbb{C}^n$; Addition in $\mathbb{C}^n$ = coordinatewise

- We use complex conjugates to define magnitudes of complex vectors.

Complex conjugate of $\vec{v}$ is $\overline{\vec{v}} = \begin{bmatrix} \overline{v_1} \\ \vdots \\ \overline{v_n} \end{bmatrix}$ (another vector in $\mathbb{C}^n$)

Note: $\overline{v_j}\, v_j = |v_j|^2$ for $j = 1, \ldots, n$

Definition: The magnitude of $\vec{v}$ is $\|\vec{v}\| = \sqrt{|v_1|^2 + \cdots |v_n|^2}$
$= \sqrt{(\overline{\vec{v}})^T \cdot \vec{v}}$

where $\vec{w}^T \cdot \vec{u} = w_1 u_1 + w_2 u_2 + \cdots + w_n u_n$ (same as with dot product in $\mathbb{R}^n$)

Note: Our definition of magnitude & dot product on $\mathbb{C}^n$ extend those from $\mathbb{R}^n$.

Example: $x = \begin{bmatrix} 2 \\ 1-i \\ 3+i \end{bmatrix}$ $y = \begin{bmatrix} i \\ 1+i \\ 2+i \end{bmatrix}$

- $x+y = \begin{bmatrix} 2 \\ 1-i \\ 3+i \end{bmatrix} + \begin{bmatrix} i \\ 1+i \\ 2+i \end{bmatrix} = \begin{bmatrix} 2+i \\ 2 \\ 5+2i \end{bmatrix}$

- $x^T \cdot y = [2, 1-i, 3+i] \begin{bmatrix} i \\ 1+i \\ 2+i \end{bmatrix} = 2i + (1-i)(1+i) + (3+i)(2+i)$
$= 2i + (1-(-1)) + (5-1) + i(5)$
$= \boxed{4+i7}$

- $\|x\| = \sqrt{\overline{x^T}\, x} = \sqrt{|2|^2 + |1-i|^2 + |3+i|^2} = \sqrt{4+2+10} = \sqrt{16} = 4$

§2 $\mathbb{C}^n$ as a $\mathbb{C}$-vector space

Structure for $\mathbb{C}^n$: Vector Space over $\mathbb{C}$ (rather than using $\mathbb{R}$ as scalars, we use $\mathbb{C}$)

- Addition: Defined coordinatewise $\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} v_1+w_1 \\ \vdots \\ v_n+w_n \end{bmatrix}$

It satisfies the same properties we had for the addition of $\mathbb{R}$-vector spaces (see Lecture 14)

(A1) $\vec{x}+\vec{y} = \vec{y}+\vec{x}$ (Commutative)

(A2) $\vec{x} + (\vec{y}+\vec{z}) = (\vec{x}+\vec{y}) + \vec{z}$ (Associative)

(A3) $\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ in $\mathbb{C}^n$ satisfies $\vec{x}+\vec{0} = \vec{0}+\vec{x} = \vec{x}$ for all $\vec{x}$ in $\mathbb{C}^n$

(A4) Given $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ in $\mathbb{C}^n$ we can find $\vec{y} = \begin{bmatrix} -x_1 \\ \vdots \\ -x_n \end{bmatrix}$ satisfying $\vec{x}+\vec{y} = \vec{0}$

- Scalar Multiplication: For each $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{C}^n$ & α in $\mathbb{C}$ (scalar) we define a new vector $\alpha \cdot \vec{v}$ in $\mathbb{C}^n$ as $\alpha \cdot \vec{v} = \begin{bmatrix} \alpha v_1 \\ \vdots \\ \alpha v_n \end{bmatrix}$

(scale each component of $\vec{v}$ by α)

Once again, we get 4 properties (same ones we had for scaling by $\mathbb{R}$)

(M1) $\alpha \cdot (\beta \cdot \vec{x}) = (\alpha\beta) \cdot \vec{x}$ [Associative]

(M2) $\alpha\cdot(\vec{x}+\vec{y}) = \alpha\cdot\vec{x} + \alpha\cdot\vec{y}$ [Distributive 1]

(M3) $(\alpha+\beta)\cdot\vec{x} = \alpha\cdot\vec{x} + \beta\cdot\vec{x}$ [Distributive 2]

(M4) $1\cdot\vec{x} = \vec{x}$ for all $\vec{x}$ in $\mathbb{C}^n$

- Can define $\mathbb{C}$-spans of vectors in $\mathbb{C}^n$ $= Sp(\vec{v}_1,\ldots,\vec{v}_p) = \{\alpha_1\vec{v}_1+\cdots+\alpha_p\vec{v}_p : \alpha_1,\ldots,\alpha_p \text{ in } \mathbb{C}\}$
- Can define $\mathbb{C}$-linear independence in $\mathbb{C}^n$: (Only solution to $\alpha_1\vec{v}_1+\cdots+\alpha_p\vec{v}_p=\vec{0}$ with $\alpha_1,\ldots,\alpha_p$ in $\mathbb{C}$ is $\alpha_1=\cdots=\alpha_p=0$)
- Can define subspaces of $\mathbb{C}^n$ (closed under $+$, $\cdot$ & $\vec{0}$ must be in the set)
- ________ basis for subspaces V (= collection of l.i. spanning vectors) & dimension of V over $\mathbb{C}$

- Same ideas will allow to build other $\mathbb{C}$-vector spaces:

① $M_{m\times n}(\mathbb{C})$ = $m\times n$ matrices with entries in $\mathbb{C}$

Addition = entry by entry, Scalar Multiplication = entry by entry

$\rightsquigarrow$ $M_{m\times n}$ will be a $\mathbb{C}$-vector space

② $P_n(\mathbb{C}) = \{P(x) = a_0 x + a_1 x + \cdots + a_n x^n : a_0, a_1, \ldots a_n \text{ in } \mathbb{C}\}$

- Polynomials in $\mathbb{C}[x]$ of degree $\leq n$
- Usual addition & scalar multiplication (now using $\mathbb{C}$)

$\rightsquigarrow$ Again: $P_n(\mathbb{C})$ will be a $\mathbb{C}$-vector space.

§3 Eigenvectors in $\mathbb{C}^n$:

Last time we saw that a matrix A of size $n\times n$ over $\mathbb{R}$ can have complex eigenvalues λ, meaning the roots of $P_A(t)$ can be non-real.

Q How do we interpret this in our original viewpoint for eigenvalues?

That is, we need to find vectors $\vec{v}\neq\vec{0}$ satisfying $A\vec{v}=\lambda\vec{v}$

Clearly, if λ is in $\mathbb{C}$, but not in $\mathbb{R}$, then $\vec{v}$ will not be in $\mathbb{R}^n$ but in $\mathbb{C}^n$

Q How do we find $\vec{v}$? Same as usual! We solve the system $(A-\lambda I_n)\begin{bmatrix} x_1\\ \vdots \\ x_n\end{bmatrix} = \begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}$ over $\mathbb{C}$.

We can use Gauss-Jordan elimination, but we are allowed to use $\mathbb{C}$

as scalars in all our elementary row operations

$\boxed{\text{EXAMPLE 1}}$ $A = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$ $\quad P_A(t) = \det\begin{pmatrix} 3-t & 1 \\ -2 & 1-t \end{pmatrix} = t^2-4t+5$

$= (t-(2+i))\,(t-(2-i))$

We get 2 eigenspaces

(1) $E_{2+i} = \text{Null Space}\,(A-(2+i)I_2) = \mathcal{N}\left(\begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix} - \begin{pmatrix} 2+i & 0 \\ 0 & 2+i \end{pmatrix}\right)$

$= \mathcal{N}\begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix}$

(2) $E_{2-i} = \text{Null Space}\,(A-(2-i)I_2) = \mathcal{N}\left(\begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix} - \begin{pmatrix} 2-i & 0 \\ 0 & 2-i \end{pmatrix}\right)$

$= \mathcal{N}\begin{pmatrix} 1+i & 1 \\ -2 & -1+i \end{pmatrix}$

- Now, we compute both Nullspaces

(1) $\begin{bmatrix} 1-i & 1 \\ -2 & -1-i \end{bmatrix} \xrightarrow[R_1 \to \frac{1}{1-i}R_1]{} \begin{bmatrix} 1 & \boxed{\frac{1}{1-i}} \\ -2 & -1-i \end{bmatrix} \xrightarrow[R_2 \to R_2+2R_1]{} \begin{bmatrix} 1 & \frac{1}{2}+i\frac{1}{2} \\ 0 & 0 \end{bmatrix}$

$= \frac{1+i}{2}$

REF $\quad$ indep var

So we need $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ to satisfy $x_1 + \left(\frac{1+i}{2}\right)x_2 = 0$

so $x_1 = \left(-\frac{1}{2} - \frac{i}{2}\right)x_2$

General form of solution: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2}-\frac{i}{2}\right)x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{-1-i}{2} \\ 1 \end{bmatrix}$

So $E_{2+i} = \left\{ x_2 \begin{bmatrix} \frac{-1-i}{2} \\ 1 \end{bmatrix} : x_2 \text{ in } \mathbb{C} \right\} = \text{Span}\left(\begin{bmatrix} \frac{-1-i}{2} \\ 1 \end{bmatrix}\right)$

__Check__: $A \begin{bmatrix} \frac{-1-i}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{-1-i}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} - i\frac{3}{2} + 1 \\ (+1+i) + 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - i\frac{3}{2} \\ 2+i \end{bmatrix}$

$(2+i)\begin{bmatrix} \frac{-1-i}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} (2+i)\left(\frac{-1-i}{2}\right) \\ 2+i \end{bmatrix} = \begin{bmatrix} \frac{-2+1+i(-2-1)}{2} \\ 2+i \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - \frac{i3}{2} \\ 2+i \end{bmatrix}$

Same!

<u>Conclusion</u> $A \begin{bmatrix} \frac{-1-i}{2} \\ 1 \end{bmatrix} = (2+i) \begin{bmatrix} \frac{-1-i}{2} \\ 1 \end{bmatrix}$

So we got the original definition of eigenvector with eigenvalue $(2+i)$

(2) $\begin{bmatrix} 1+i & 1 \\ -2 & -1+i \end{bmatrix} \xrightarrow[R_1 \to (1-i)R_1]{} \begin{bmatrix} (1+i)(1-i) & 1-i \\ -2 & -1+i \end{bmatrix} = \begin{bmatrix} 1+1 & 1-i \\ -2 & -1+i \end{bmatrix}$
(*)

$\xrightarrow[R_2 \to R_2 + R_1]{} \begin{bmatrix} 2 & 1-i \\ 0 & 0 \end{bmatrix} \xrightarrow[R_1 \to \frac{R_1}{2}]{} \begin{bmatrix} 1 & \frac{1-i}{2} \\ 0 & 0 \end{bmatrix}$

indep

(*) We multiplied R_1 by the complex conjugate of the (1,1)-entry, to turn it into a positive real number.

Once more $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in $\mathbb{C}^n$ must solve $x_1 + \left(\frac{1-i}{2}\right) x_2 = 0$

$$\rightsquigarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\left(\frac{1-i}{2}\right) x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{-1+i}{2} \\ 1 \end{bmatrix}$$

$$E_{2-i} = \left\{ x_2 \begin{bmatrix} \frac{-1+i}{2} \\ 1 \end{bmatrix} : x_2 \text{ in } \mathbb{C} \right\} = \operatorname{Span}\left(\begin{bmatrix} \frac{-1+i}{2} \\ 1 \end{bmatrix} \right)$$

Can check that $A \begin{bmatrix} \frac{-1+i}{2} \\ 1 \end{bmatrix} = (2-i) \begin{bmatrix} \frac{-1+i}{2} \\ 1 \end{bmatrix}$.

<u>Observe</u>: Got $E_{2+i} = \operatorname{Sp}\left(\begin{bmatrix} \frac{-1-i}{2} \\ 1 \end{bmatrix} \right)$ & $E_{2-i} = \operatorname{Sp}\left(\begin{bmatrix} \frac{-1+i}{2} \\ 1 \end{bmatrix} \right)$

$2+i$ & $2-i$ are complex conjugates & so are the generating vectors of the 2 eigenspaces!

This is true in general!

<u>Proposition</u>: If A has size $n \times n$ & entries in $\mathbb{R}$, & λ is an eigenvalue of A in $\mathbb{C}$ but not in $\mathbb{R}$ with eigenvector $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{C}^n$ then $\bar{\lambda}$ is an eigenvalue of A & $\bar{v} = \begin{bmatrix} \bar{v}_1 \\ \vdots \\ \bar{v}_n \end{bmatrix}$ in $\mathbb{C}^n$ is an eigenvector for $\bar{\lambda}$.

Why? Say $A \cdot \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ write $A = \begin{bmatrix} a_{11} & & a_{1n} \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$

$$\begin{bmatrix} a_{11}v_1 + \cdots + a_{1n}v_n \\ a_{21}v_1 + \cdots + a_{2n}v_n \\ \vdots \\ a_{n1}v_1 + \cdots + a_{nn}v_n \end{bmatrix} = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \\ \vdots \\ \lambda v_n \end{bmatrix}$$

Now, conjugate both vectors & use all a_{jk} are in $\mathbb{R}$.

$$A\overline{v} = \begin{bmatrix} a_{11}\overline{v_1} + \cdots + a_{1n}\overline{v_n} \\ \vdots \\ a_{n1}\overline{v_1} + \cdots + a_{nn}\overline{v_n} \end{bmatrix} = \begin{bmatrix} \overline{\lambda}\,\overline{v_1} \\ \vdots \\ \overline{\lambda}\,\overline{v_n} \end{bmatrix} = \overline{\lambda} \begin{bmatrix} \overline{v_1} \\ \vdots \\ \overline{v_n} \end{bmatrix}$$

and since $v \neq \vec{0}$, we get $\overline{v} \neq \vec{0}$ so it's an eigenvector for $\overline{\lambda}$.
& $\overline{\lambda}$ is an eigenvalue for A. $\square$

- We can do more! We can compute basis for $E_{\overline{\lambda}}$ using one for E_λ.

Proposition 2: If λ is a non-real eigenvalue for A of size $n \times n$ with entries in $\mathbb{R}$ and $B = \{v_1, \ldots v_p\}$ is a basis for E_λ (that is l.i. set that spans E_λ)
then, $\overline{B} = \{\overline{v_1}, \ldots \overline{v_p}\}$ is a basis for $E_{\overline{\lambda}}$.

Proof: By Proposition 1 we know $\overline{v_1}, \ldots \overline{v_p}$ all lie in $E_{\overline{\lambda}}$.

Claim 1: If $\{v_1, \ldots, v_p\}$ is l.i., so is $\{\overline{v_1}, \ldots \overline{v_p}\}$.

Why? Want to solve $\alpha_1 \overline{v_1} + \cdots + \alpha_p \overline{v_p} = \vec{0}$

If we conjugate both sides we get

$$(*) \quad \overline{\alpha_1}\,\overline{\overline{v_1}} + \cdots + \overline{\alpha_p}\,\overline{\overline{v_p}} = \overline{0} = 0$$

But $\overline{\overline{v_1}} = v_1, \ldots, \overline{\overline{v_p}} = v_p$ (because $\overline{\overline{a+ib}} = \overline{a-ib} = a+ib$)

So $(*)$ becomes $\overline{\alpha_1} v_1 + \cdots + \overline{\alpha_p} v_p = 0$.

But since $\{v_1, \ldots, v_p\}$ is l.i., we conclude $\overline{\alpha_1} = \overline{\alpha_2} = \cdots = \overline{\alpha_p} = 0$

but then
$$\alpha_1 = \overline{\overline{\alpha_1}} = \overline{0} = 0$$
$$\alpha_2 = \overline{\overline{\alpha_2}} = \overline{0} = 0$$
$$\vdots$$
$$\alpha_p = \overline{\overline{\alpha_p}} = \overline{0} = 0$$

Conclusion: Our original system $\alpha_1 \overline{v_1} + \cdots + \alpha_p \overline{v_p} = 0$ only has one solution, the trivial one. $\alpha_1 = \cdots = \alpha_p = 0$.

- In particular $\boxed{\dim E_\lambda \leq \dim E_{\overline{\lambda}}}$ (1)

- But we can reverse the process: start from a basis B' for $E_{\overline{\lambda}}$ & conjugate its elements to get a linearly independent set $\overline{B'}$ in E_λ.

So $\boxed{\dim E_{\overline{\lambda}} \leq \dim E_\lambda}$ (2)

Comparing (1) & (2) we get $\dim E_{\overline{\lambda}} = \dim E_\lambda$ & the claim about linear independence will give us the claim for bases

(Why? Just as it happened for subspaces of $\mathbb{R}^n$, if we have a subspace W of $\mathbb{C}^n$ with $\dim W = p$ & a collection $\{v_1, \ldots, v_p\}$ of vectors in W that are l.i., they automatically become a basis for W because we have the right number of vectors = as many as $\dim W$) $\square$

Q: Why is this an important result?
A: It will save us a lot of time when computing bases for eigenspaces, since we'll need to compute only one of E_λ or $E_{\overline{\lambda}}$.

§3 Special case: Symmetric matrices

There is one very special situation, when we know all eigenvalues are real. This is the case when the input matrix A is symmetric

Key Theorem: If A is an $n \times n$ real symmetric matrix, then all its eigenvalues are real! (Next time: we'll see we have a basis of eigenvectors)

Proof: Assume λ is an eigenvalue of A with eigenvector $\vec{0} \neq v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{C}^n$

want to show λ is in $\mathbb{R}$. We'll show $\lambda = \bar{\lambda}$.

How? Start from $\boxed{Av = \lambda v}$ (*) & multiply both sides on the left by $\bar{v}^T$

$$\bar{v}^T A v = \bar{v}^T \lambda v = \lambda \bar{v}^T v = \lambda \|v\|^2 \quad (1)$$

But $\bar{v}^T (\lambda v) = [\bar{v}_1, \dots \bar{v}_n] \begin{bmatrix} \lambda v_1 \\ \vdots \\ \lambda v_n \end{bmatrix} = [\lambda v_1 \cdots \lambda v_n] \begin{bmatrix} \bar{v}_1 \\ \vdots \\ \bar{v}_n \end{bmatrix} = (\lambda v)^T \bar{v}$

So $\bar{v}^T(\lambda v) = (\lambda v)^T \bar{v} \underset{\text{by } (*)}{=} (Av)^T \bar{v} = (v^T A^T) \bar{v} \underset{A \text{ symmetric}}{=} (v^T A) \bar{v} =$

$$\underset{\text{Assoc.}}{=} v^T (A\bar{v}) \underset{\text{Prop 1}}{=} v^T (\bar{\lambda} \bar{v}) \quad (2)$$

Once again: $v^T (\bar{\lambda}\bar{v}) = [v_1 \cdots v_n] \begin{bmatrix} \bar{\lambda}\bar{v}_1 \\ \vdots \\ \bar{\lambda}\bar{v}_n \end{bmatrix} = [\bar{\lambda}\bar{v}_1, \dots, \bar{\lambda}\bar{v}_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

$$= \bar{\lambda} \bar{v}^T v = \bar{\lambda} \|v\|^2 \quad (3)$$

Gathering (1), (2) & (3) we get:

$$\lambda \|v\|^2 = \bar{v}^T \lambda v = v^T (\bar{\lambda}\bar{v}) = \bar{\lambda} \|v\|^2$$

So $\lambda \|v\|^2 - \bar{\lambda} \|v\|^2 = (\lambda - \bar{\lambda}) \|v\|^2 = 0$

But $v \neq \vec{0}$ forces $\|v\|^2 \neq 0$

Then $(\lambda - \bar{\lambda}) \|v\|^2 = 0$ can only happen if $\lambda - \bar{\lambda} = 0$, that is $\lambda = \bar{\lambda}$.

Conclusion: λ is in $\mathbb{R}$.

Example: $A = \underset{\text{symmetric}}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \rightsquigarrow P_A(t) = \det \begin{bmatrix} -t & 1 \\ 1 & -t \end{bmatrix} = t^2 - 1 = (t+1)(t-1)$

A has 2 distinct eigenvalues, so by a Theorem from Lecture 33 we know A has a basis of eigenvectors

$E_1 = Sp\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$, $E_{-1} = Sp\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) \rightsquigarrow$ Basis for $\mathbb{R}^2$ $= \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$