Lecture 36: §4.7 Similarity & Diagonalization

Last time: • Defined the $\mathbb{C}$-vector space $\mathbb{C}^n$
• Studied complex eigenvectors of matrices of size $n\times n$ with real entries
• Real $n\times n$ symmetric matrices have ONLY real eigenvalues.

TODAY: Study the notion of similarity & diagonalization of matrices, that is, to find a basis of eigenvectors for $\mathbb{R}^n$. Main example: real symmetric $n\times n$ matrices

§1. Similarity:

Definition: Two matrices A & C of size $n\times n$ with complex entries are similar if there is a nonsingular $n\times n$ matrix S with complex entries such that $C = S^{-1}AS$

• For the applications we have in mind, we will only consider the case where A, C & S have real entries, but the definition is more general.

Motivation: A & C will be representing matrices of the SAME linear transformation $T:\mathbb{R}^n \longrightarrow \mathbb{R}^n$ with respect to two different bases: B_A & B_C & S will be the matrix recording the "change of basis" from B_C to B_A.
Typical situation: $T(\vec{v}) = A\vec{v}$ so $B_A = E = \{e_1, \ldots e_n\}$ canonical basis for $\mathbb{R}^n$
If $B_C = \{\vec{v}_1, \ldots \vec{v}_n\}$, then $S = [v_1 \cdots v_n]$ (columns = vectors in the basis B_C)

• We will show that similar matrices have the same set of eigenvalues (with the same algebraic multiplicity). To this end, we show

Proposition 1: Similar matrices have the same characteristic polynomial

Proof: ($C=S^{-1}AS$) $\boxed{P_C(t)} = \det(S^{-1}AS - tI_n) = \det(S^{-1}(A-tI_n)S)$
$= \det(S^{-1})\det(A-tI_n)\det(S) \overset{\det S\neq 0}{=} \frac{1}{\det(S)} P_A(t)\det(S) = \boxed{P_A(t)}$

Remark: If $P_A(t) = P_B(t)$ this does not necessarily mean A & B are similar

Example: $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ & $B = I_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ both have characteristic polynomial $(t-1)^2$

But B is only similar to itself! (Because no matter what non-singular matrix S we pick, we get $S^{-1}\boxed{I_2}S = S^{-1}S = \boxed{I_2}$)

Q: If A & C are similar & λ is an eigenvalue of both, how are the eigenspaces $E_\lambda(A)$ & $E_\lambda(C)$ related?

The next result will help us answer that question for $C = S^{-1}AS$

Proposition 2: If $\vec{v}$ is an eigenvector of C with eigenvalue λ, then $\vec{w} = S\vec{v}$ is an eigenvector of A with eigenvalue λ.

Proof: Use the definition of eigenvectors.

- First $\vec{v} \neq \vec{0}$ & S is singular (that is, $N(S) = \{0\}$), so $\vec{w} = S\vec{v}$ is also $\neq \vec{0}$.
- Now, $A\vec{w} = AS\vec{v} = \overbrace{S(\underbrace{S^{-1}AS}_{=C}}^{=I_n})\vec{v}$

$$= S(C\vec{v}) \underset{\vec{v}\text{ eigenvector}}{=} S\lambda\vec{v} = \lambda(S\vec{v}) = \lambda\vec{w}$$

Conclude: $A\vec{w} = \lambda\vec{w}$ & $\vec{w} \neq \vec{0}$ so it's an eigenvector of A with eigenvalue λ.

Obs: We can switch the roles of A & C to get the following (writing $(S^{-1})^{-1}C(S^{-1}) = A$)

Corollary 1: If $\vec{w}$ is an eigenvector of A with eigenvalue λ, then $\vec{v} = S^{-1}\vec{w}$ is an eigenvector of C with eigenvalue λ

Combining Prop 2 & Corollary 1, we get an invertible linear transf between $E_\lambda(A)$ & $E_\lambda(C)$, namely

$$T: E_\lambda(C) \longrightarrow E_\lambda(A)$$
$$\vec{v} \longmapsto S\vec{v}$$

$$T^{-1}: E_\lambda(A) \longrightarrow E_\lambda(C)$$
$$\vec{w} \longmapsto S^{-1}\vec{w}$$

Consequences: ① $\dim E_\lambda(C) = \dim E_\lambda(A)$

② If $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is a basis for $E_\lambda(C)$, then $\{S\vec{v}_1, \ldots, S\vec{v}_p\}$ is a basis for $E_\lambda(A)$.

Example: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ We claim: A & C are similar matrices

- $p_A(t) = \det \begin{vmatrix} -t & 1 \\ 1 & -t \end{vmatrix} = t^2 - 1 = (t+1)(t-1) = p_C(t)$
- $E_1(A) = \mathcal{N}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\right)$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \xrightarrow[R_2 \to R_2 + R_1]{} \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \xrightarrow[R_1 \to -R_1]{} \underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}}_{\text{REF}} \rightsquigarrow x_1 - x_2 = 0 \rightsquigarrow \boxed{x_1 = x_2}$$

indep.

General soln: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightsquigarrow E_1(A) = \mathrm{Sp}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

- $E_{-1}(A) = \mathcal{N}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)$

indep

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow[R_2 \to R_2 - R_1]{} \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{\text{REF}} \rightsquigarrow x_1 + x_2 = 0 \quad \boxed{x_1 = -x_2}$$

General soln $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

So $E_{-1}(A) = \mathrm{Sp}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$

Claim: If $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\vec{v} \longmapsto A\vec{v}$, then if $\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

we have $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ & $T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = -\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ so $[T]_{\mathcal{B}\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = C$

Then $A = [T]_{EE}$ & we can write $C = S^{-1}AS$

with $S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ (columns = basis B where $[T]_{BB} = C$)

<u>Let's check this!</u> $S^{-1} \underset{\substack{\uparrow \\ 2\times 2 \text{ formula}}}{=} \frac{1}{\det(S)} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$$S^{-1}AS = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = C \checkmark$$

<u>Summary</u>:
- We showed A & C are similar by explicitly computing the matrix S. For this, we used eigenvectors of A <u>because</u> C was a diagonal matrix.
- This example shows how the "diagonalization" process will work.

<u>§2 Diagonalization</u>:

<u>Definition</u>: We say an $n\times n$ real matrix A is <u>diagonalizable</u> (over $\mathbb{C}$) if it is similar to a diagonal matrix $D = \begin{pmatrix} \boxed{d_1} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \boxed{d_n} \end{pmatrix}$ (with entries in $\mathbb{C}$)

(If we work over $\mathbb{C}$, the matrix S also has entries in $\mathbb{C}$)

<u>Observe</u>: $P_D(t) = \det(D - tI_n) = \det\begin{pmatrix} d_1 - t & & 0 \\ & \ddots & \\ 0 & & d_n - t \end{pmatrix} = (d_1 - t)\cdots(d_n - t)$

So the eigenvalues of D are $d_1, \ldots, d_n$ (counted with multiplicity, that is, we can have repetitions!)

Since similar matrices have the same characteristic polynomials, we conclude that the entries along the diagonal in D <u>are</u> the eigenvalues of A.

Q: Why is it useful to diagonalize a matrix (whenever possible)?

<u>A</u>: We can perform power operations very fast!

Indeed, if $D = S^{-1}AS$, then $D^k = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}^k = \begin{pmatrix} d_1^k & & 0 \\ & \ddots & \\ 0 & & d_n^k \end{pmatrix}$

$$D^k = (S^{-1}AS)^k = \underbrace{(S^{-1}A\underbrace{S)(S^{-1}}_{=I_n}AS) \cdots (S^{-1}A\underbrace{S)(S^{-1}}_{I_n}AS)}_{k \text{ times}}$$

so the middle terms all cancel out & we get

$$D^k = S^{-1}A^kS$$

So $S\boxed{D^k}S^{-1} = \underbrace{S(S^{-1}}_{=I_2}A^k\underbrace{S)S^{-1}}_{I_2} = A^k$

easy to do!

Conclude: $A^k = SD^kS^{-1} = S\begin{pmatrix} d_1^k & & 0 \\ & \ddots & \\ 0 & & d_n^k \end{pmatrix}S^{-1}$ for all $k = 1, 2, 3, \ldots$

Our next result characterizes when a matrix A is diagonalizable over $\mathbb{C}$. As we saw in the example on page 3, we must have a basis for $\mathbb{C}^n$ consisting of eigenvectors of A (over $\mathbb{C}^n$). We distinguish the complex vs real setting.

Theorem ① An $n \times n$ matrix A is diagonalizable over $\mathbb{C}$ if and only if A has a set of n linearly independent eigenvectors in $\mathbb{C}^n$

② An $n \times n$ matrix A with real entries is diagonalizable over $\mathbb{R}$ if all eigenvalues of A are real & we have n linearly independent eigenvectors of A in $\mathbb{R}^n$

Proof: We discuss ①. For statement ②, it suffices to notice that D will be the same in ① & ②, so D being real means the eigenvalues of A are all real. Once this is established, we see that to compute $E_\lambda(A)$ for λ real & A with real entries we never leave the world of real (as oppose to complex) entries. So the basis of eigenvectors in ① is actually in $\mathbb{R}^n$! This shows ② follows from ①.

We prove ① by showing the two implications

$\boxed{\text{One direction:}}$ Assume we have a basis $B = \{v_1 \cdots v_n\}$ for $\mathbb{C}^n$ consisting of eigenvectors of A. Write $S = [v_1 \cdots v_n]$

- S is non-singular (as a matrix with complex entries) because the columns are li. In particular, S is invertible (again, over $\mathbb{C}$)
- By definition, we know $Av_i = \lambda_i v_i$ for λ_i eigenvalue of A
- We want to show: $S^{-1}AS = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

$$AS = A[v_1 \cdots v_n] = [\lambda_1 v_1 \cdots \lambda_n v_n]$$

So $S^{-1}AS = S^{-1}[\lambda_1 v_1 \cdots \lambda_n v_n] = [\lambda_1 S^{-1} v_1 \cdots \lambda_n S^{-1} v_n]$

$$S^{-1} v_j = S^{-1}(\mathrm{col}_j S) = e_j \left(= \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j^{th} \text{ spot} \right) \text{ because } S^{-1}S = I_n$$

So $S^{-1}AS = [\lambda_1 e_1 \cdots \lambda_n e_n] = \begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & & \lambda_n \end{bmatrix}$ is diagonal! ✓

$\boxed{\text{Other direction:}}$ Assume A is diagonalizable & write

$$D = S^{-1}AS \text{ for some invertible matrix } S \text{ \& } D = \text{diagonal}$$

Then $SD = AS$

- Write the column vectors of S as $\{v_1 \cdots v_n\}$ (they are vectors in $\mathbb{C}^n$)
- Write $D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$

Then $SD = [v_1 \cdots v_n]\begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} = [d_1 v_1, \cdots d_n v_n]$ $(*)$

(For example $[v_1 \cdots v_n]\begin{bmatrix} d_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = d_1 [v_1 \cdots v_n]\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = d_1 v_1$, the others follow by the same reasoning)

Now: $AS = [Av_1, \cdots Av_n]$ $(**)$

Putting $(*)$ & $(**)$ together & comparing column by column gives

$$\boxed{d_1 \vec{v}_1 = A\vec{v}_1, \quad d_2 \vec{v}_2 = A\vec{v}_2, \quad \ldots, \quad d_n \vec{v}_n = A\vec{v}_n} \quad (***)$$

Since S is nonsingular $\{\vec{v}_1, \ldots \vec{v}_n\}$ is a basis for $\mathbb{C}^n$. In particular, all $\vec{v}_i$'s are $\neq \vec{0}$ & by $(***)$ they are eigenvectors of A. ✓

Remark: It is important to notice that the basis of eigenvectors is ORDERED by the location of the eigenvalues placed along the diagonal of D.

Example 1: $A = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$ $\quad P_A(t) = \det\begin{pmatrix} 5-t & -6 \\ 3 & -4-t \end{pmatrix} = (5-t)(-4-t)+18$
$= t^2 - t + 2 = (t-2)(t+1)$

Since we have 2 distinct & real eigenvalues we know from Lecture 33 (Consequence 1, page 5), that A has a basis of eigenvectors for $\mathbb{R}^2$, so it's diagonalizable over $\mathbb{R}$.

E_{-1}: Solve $\begin{bmatrix} 6 & -6 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow x_1 - x_2 = 0$ so $\boxed{x_1 = x_2}$ $\quad E_{-1} = Sp\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

E_2: Solve $\begin{bmatrix} 3 & -6 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow x_1 - 2x_2 = 0$ so $x_1 = 2x_2$ $\quad E_2 = Sp\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \quad \& \quad S = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \rightsquigarrow S^{-1} = \frac{1}{\underbrace{\det S}_{=-1}}\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

basis for E_{-1} $\qquad$ basis for E_2

Easy Check: $D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \overset{?}{=} S^{-1}AS = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ ✓

Example 2: Use this to compute A^{10}

$$A^{10} = S D^{10} S^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} (-1)^{10} & 0 \\ 0 & 2^{10} \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1024 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1024 & -1024 \end{bmatrix} = \begin{bmatrix} 2047 & -2046 \\ 1023 & -1022 \end{bmatrix}$$

Example 3 Let's look at Example 1 on page 4 of Lecture 35.

$A = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$ $\quad$ eigenvalues $= 2+i,\ 2-i \rightsquigarrow$ so diagonalizable over $\mathbb{C}^2$.

$$E_{2+i} = Sp\left(\begin{bmatrix} -1-i \\ 2 \end{bmatrix}\right) \quad \& \quad E_{2-i} = Sp\left(\overline{\begin{bmatrix} -1-i \\ 2 \end{bmatrix}}\right) = Sp\left(\begin{bmatrix} -1+i \\ 2 \end{bmatrix}\right)$$

$$S = \begin{bmatrix} -1-i & -1+i \\ 2 & 2 \end{bmatrix} \quad \text{for } D = \begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix}$$

To compute S^{-1}, we use the usual formula for 2×2 matrices (it works for both real & complex matrices!) $\det S = (-1-i)2 - (-1+i)2$
$= -2-i2+2-i2 = -4i$

$$S^{-1} = \frac{1}{\det(S)} \begin{bmatrix} 2 & -(-1+i) \\ -2 & -1-i \end{bmatrix} = \boxed{\frac{1}{-4i}} \begin{bmatrix} 2 & 1-i \\ -2 & -1-i \end{bmatrix} = \begin{bmatrix} i/2 & \frac{1+i}{4} \\ -i/2 & \frac{1-i}{4} \end{bmatrix}$$

$$= \frac{\overline{(-4i)}}{|-4i|^2} = \frac{4i}{16} = \frac{i}{4}$$

<u>Check</u> $D = S^{-1} A S$

$$\begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} i/2 & \frac{1+i}{4} \\ -i/2 & \frac{1-i}{4} \end{bmatrix} \underbrace{\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -1-i & -1+i \\ 2 & 2 \end{bmatrix}}_{\begin{bmatrix} (-3+2)-i3 & (-3+2)+i3 \\ (2+2)+i2 & (2+2)-i2 \end{bmatrix} = \begin{bmatrix} -1-i3 & -1+i3 \\ 4+i2 & 4-i2 \end{bmatrix}}$$

Compare all 4 entries:

$$2+i \stackrel{?}{=} \frac{i}{2}(-1-i3) + (4+i2)\left(\frac{1+i}{4}\right) = \frac{3}{2} - \frac{i}{2} + \frac{1}{2}\left(2-1+i(2+1)\right) ✓$$

$$0 \stackrel{?}{=} \frac{i}{2}(-1+i3) + (4-i2)\left(\frac{1+i}{4}\right) = \frac{-3}{2} - \frac{i}{2} + \frac{1}{2}\left(2+1+i(2-1)\right) ✓$$

$$0 \stackrel{?}{=} -\frac{i}{2}(-1-i3) + (4+i2)\left(\frac{1-i}{4}\right) = \frac{-3}{2} + \frac{i}{2} + \frac{1}{2}\left(2+1+i(-2+1)\right) ✓$$

$$2-i \stackrel{?}{=} -\frac{i}{2}(-1+i3) + (4-i2)\left(\frac{1-i}{4}\right) = \frac{3}{2} + \frac{i}{2} + \frac{1}{2}\left(2-1+i(-2-1)\right) ✓$$

<u>Q</u>: What about other sizes?

In Lecture 33 Thm page 7, we saw that eigenvectors with different eigenvalues are l.i. This has a powerful consequence:

<u>Theorem</u> Fix $B_\lambda = \{\vec{v}_1, \ldots \vec{v}_p\}$ a basis for the eigenspace E_λ of a matrix A
$B_\mu = \{\vec{w}_1, \ldots \vec{w}_q\}$ ——— E_μ ———

Then $B_\lambda \cup B_\mu = \{\vec{v}_1, \ldots \vec{v}_p, \vec{w}_1, \ldots \vec{w}_q\}$ is linearly indep.

Proof: Write $\left(\underbrace{\alpha_1\vec{v}_1+\dots+\alpha_p\vec{v}_p}_{\text{it's an eigenvector in } E_\lambda}\right)+\left(\underbrace{\beta_1\vec{w}_1+\dots+\beta_q\vec{w}_q}_{\text{it's an eigenvector in } E_\mu}\right)=\vec{0}$

$$\vec{v}=\alpha_1\vec{v}_1+\dots+\alpha_p\vec{v}_p = -(\beta_1\vec{w}_1+\dots+\beta_q\vec{w}_q)=-\vec{w}$$

So
$$\left.\begin{array}{l} A\vec{v}=\lambda\vec{v} \\ \quad \| \\ A(-\vec{w})=\mu(-\vec{w})=\mu\vec{v}\end{array}\right\}\quad (\lambda-\mu)\vec{v}=\vec{0} \ \&\ \lambda\neq\mu \text{ forces } \vec{v}=\vec{0},\ \vec{w}=\vec{0}$$

But then $\vec{0}=\alpha_1\vec{v}_1+\dots+\alpha_p\vec{v}_p$ & li so $\alpha_1=\dots=\alpha_p=0$

$\vec{0}=\beta_1\vec{v}_1+\dots+\beta_q\vec{v}_q$ ——— $\beta_1=\dots=\beta_q=0$

Conclude $\{\vec{v}_1,\dots,\vec{v}_p,\vec{\beta}_1,\dots,\vec{\beta}_q\}$ are li. □

Obs: The same idea works if we have more than 2 eigenspaces.

Theorem: A $n\times n$ with eigenvalues $\lambda_1,\dots,\lambda_s$ (counted with multiplicity)

Then: A is diagonalizable over $\mathbb{C}$ if and only if
$$\dim E_{\lambda_1}+\dots+\dim E_{\lambda_s}=n$$

(This will ensure that we have a basis of eigenvectors $= \mathcal{B}_{\lambda_1}\cup\mathcal{B}_{\lambda_2}\cup\dots\cup\mathcal{B}_{\lambda_s}$
(just collect the basis for each E_{λ_i})

Example 4

$$A=\begin{bmatrix}25 & -8 & 30\\ 24 & -7 & 30\\ -12 & 4 & -14\end{bmatrix} \rightsquigarrow p_A(t)=-t^3+4t^2-5t+2 = (1-t)^2(2-t)$$

We compute the 2 eigenspaces:

$$\left.\begin{array}{l} E_1 = \mathcal{N}\begin{pmatrix}24 & -8 & 30\\ 24 & -8 & 30\\ -12 & 4 & -15\end{pmatrix} = \mathrm{Sp}\left(\begin{bmatrix}1\\3\\0\end{bmatrix},\begin{bmatrix}-4\\3\\4\end{bmatrix}\right) \rightsquigarrow \dim = 2 \\ E_2 = \mathcal{N}\begin{pmatrix}23 & -8 & 30\\ 24 & -9 & 30\\ -12 & 4 & -16\end{pmatrix} = \mathrm{Sp}\left(\begin{bmatrix}-2\\-2\\1\end{bmatrix}\right) \rightsquigarrow \dim = 1\end{array}\right\}\ 2+1=3 \text{ so } A \text{ is diagonalizable}$$

In fact, we have $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ & $S = \begin{bmatrix} 1 & -4 & -2 \\ 3 & 3 & -2 \\ 0 & 4 & 1 \end{bmatrix}$

§3 Real Symmetric Matrices:

In addition to having only real eigenvalues, real symmetric matrices are always diagonalizable. Furthermore, we can choose an orthonormal basis of eigenvectors. Here's the statement, followed by an example:

Theorem Fix a real $n \times n$ symmetric matrix A. Then

(1) We can find a basis $\{\vec{v}_1, \dots, \vec{v}_n\}$ of eigenvectors for $\mathbb{R}^n$ that is orthonormal, so $\vec{v}_i \cdot \vec{v}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$

(2) If $Q = [v_1 \cdots v_n]$, then $Q^{-1} = Q^T$ & $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} = Q^T A Q$

Example: $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ $\qquad P_A(t) = (1-t)^2 - 1 = t(t-2)$

$E_0 = Sp\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

$B = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$ is orthogonal, but norms are not 1, so we need to adjust for this.

$E_2 = Sp\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$

$\left\|\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\| = \sqrt{2}$ & $\left\|\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\| = +\sqrt{2}$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Check $Q^T Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ as expected!

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$