Lecture 1 : §1.1 Matrices & Systems of Linear Equations

Two Objectives

① Solve the simplest type of equations in 2 or more variables
(LINEAR!) (MATH 2153 - Calc III)

Example: $x + 2y + z = 10$ describes a plane in $\mathbb{R}^3$ with normal direction $\langle 1,2,1 \rangle$, passing through $(10,0,0)$

② Introduce you to the underlying algebraic structures of these solution sets ⇝ Vector spaces (of finite dimension)

Example: Solutions to homogeneous differential equations, such $y'' + y = 0$

- $y_{(x)} = a \boxed{\cos(x)} + b \boxed{\sin(x)}$ for any a, b real number
  basis for the space of solutions
- Can add two solutions & get another solution
- Can multiply a solution by a fixed number & get a soln

"Plane of solutions"

Definition: A linear equation in n unknowns (or variables) is an equation of the form:

$$\boxed{a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b}$$

- $a_1, a_2, \ldots, a_n$ are the coefficients (fixed numbers in $\mathbb{R}, \mathbb{Q}, \mathbb{C}, \ldots$)
- b is the constant term or constant coefficient (also fixed!)
- $x_1, \ldots, x_n$ are the unknowns (variables)

Examples:

$$x_1 - 4x_2 + x_3 - x_4 = 10$$

$$x + 2y + z = 10$$

(usually for $n=3$ we use x, y, z as variable names)

Q: Why linear?

A: Each variable has exponent 1

Non-example: $x_1 + 2\sin(x_2) = 0$ is linear in x_1 but non-linear on x_2

Def: A <u>solution</u> to $\boxed{a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b}$ is a tuple of numbers $(x_1, x_2, \ldots, x_n)$ verifying this eqn.

Example: $(1, 5, -1)$ is a solution to $X + 2Y + Z = 10$

$(0, 0, 10)$ is also a solution.

Objective: Study solutions to a group (or system) of linear eqns

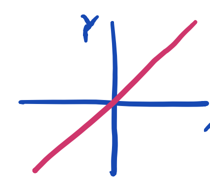
↳ • Estimate their number

• Find all solutions (Algorithm?)

• Write them down!

$\boxed{\text{First main result}}$: Classification of the solutions sets of linear systems

3 options: ONLY

1. no solutions $\left(\begin{cases} X + Y = 0 \\ X + Y = 1 \end{cases} \right)$
2. unique solution $\left(\begin{cases} X + Y = 2 \\ X - Y = 0 \end{cases} \text{ only soln: } \begin{matrix} X = 1 \\ Y = 1 \end{matrix} \right)$
3. infinitely many solutions $\left(X - Y = 0 \right.$  $\left. \right)$

§2 Linear systems

GOAL: Find simultaneous (or joint) solutions to several linear equations in an algorithmic way (GAUSS-JORDAN ELIMINATION)

EXAMPLES: ① $\begin{cases} 2X + 4Y = 18 \\ X - Y = 0 \end{cases}$ 2 equations, 2 unknowns

Method 1: Manipulate the equations to reduce the number of terms in each one, whenever possible. Solve one var at a time.

- Multiply Eqn 2 by 4 & add it to Eqn 1:

$$\begin{cases} 2X + 4Y = 18 \\ 4X - 4Y = 0 \end{cases} \longrightarrow \begin{cases} 2X + 4Y = 18 \\ 6X = 18 \end{cases} \rightsquigarrow \boxed{X = \frac{18}{6} = 3}$$

unique solution!

- Substitute value for X in Eqn 1 & solve for Y

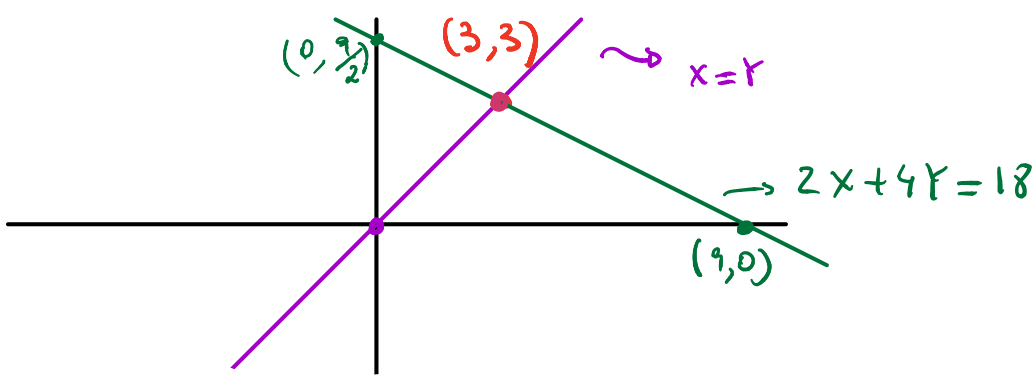
$$2 \cdot 3 + 4Y = 18 \rightsquigarrow 4Y = 18 - 6 = 12 \rightsquigarrow \boxed{Y = 3}$$

$$\begin{cases} 2x + 4y = 18 \\ x - y = 0 \end{cases}$$

$\boxed{\text{Method 2:}}$ Use Geometry!

- Each equation represents a line in the plane
- Draw the lines & see pts where they meet: that's the solution to the system!

Obs: The lines are not parallel, so we know they meet in a single point (⇝ unique solution)



② $\begin{cases} 2X + 4Y + Z = 18 \\ X - Y = 0 \end{cases}$

2 equations
3 unknowns

Method 1   Start from Eqn 2 & replace $Y = X$ in 1st one & solve for Z

$$\begin{cases} 2X + 4Y + Z = 18 \\ X - Y = 0 \end{cases} \rightsquigarrow \begin{cases} \overbrace{2X + 4X}^{=6X} + Z = 18 \\ Y = X \end{cases} \rightsquigarrow \begin{cases} 6X + Z = 18 \\ Y = X \end{cases}$$

$$\rightsquigarrow \begin{cases} Z = 18 - 6X \\ Y = X \end{cases}$$

X has no restrictions (free parameter)

Conclude: We have infinitely many solutions

$X = 0$ gives $(0, 0, 18)$

$X = 1$ gives $(1, 1, 12)$

Parametric form $= (X, X, 18 - 6X) = (0, 0, 18) + \underline{\underline{X}}(1, 1, -6)$

A: Line in 3-space with direction $\langle 1, 1, -6 \rangle$ passing through $(0, 0, 18)$

$$\begin{cases} 2X + 4Y + Z = 18 \\ X - Y = 0 \end{cases}$$

Method 2: Think of each equation as a plane in 3 space

Plane 1: Normal direction $\vec{n}_1 = \langle 2, 4, 1 \rangle$

Plane 2 ———————— $\vec{n}_2 = \langle 1, -1, 0 \rangle$

not parallel!

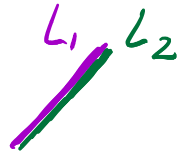
So the planes are not parallel. Then, they MUST meet along a line (this is our solution!)

- Direction of line: $\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 2 & 4 & 1 \\ 1 & -1 & 0 \end{vmatrix} = \langle 1, 1, -6 \rangle$
- Point $(0, 0, 18)$ (Easy guess)

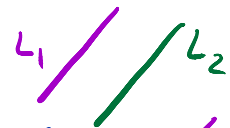
Obs: We got the same using method 1 $\rightsquigarrow$ more algorithmic.

SUMMARY: ① 2 equations & 2 unknowns = intersection of 2 lines in $\mathbb{R}^2$.

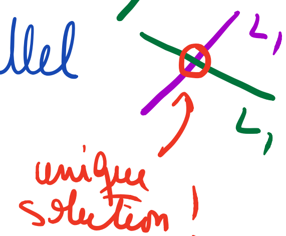
Three possibilities:

(i) 2 lines coincide 

= infinitely many solutions

(ii) 2 lines are different but parallel 

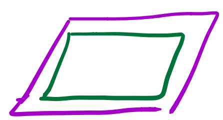
= no soln

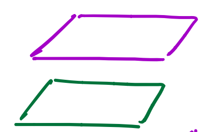
(iii) ______________________ & are not parallel 

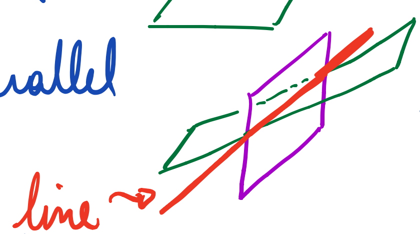
= unique solution

② 2 equations & 3 unknowns = intersection of 2 planes in $\mathbb{R}^3$.

Three options:

(i) 2 planes coincide  = infinitely many solutions

(ii) 2 planes are different but parallel  = no solution

(iii) ______________________ & not parallel 

= line of solutions

⚠ More unknowns than equations will never give a unique soln.