Lecture 2: §1.1 Matrices & Systems of Linear Equations II
§1.2 Echelon forms

Last Time:
- Overview of course
- $\boxed{\text{Main result:}}$ systems of linear equations have either
  1. no solution
  2. a unique solution
  3. infinite many solutions
- Saw 2 methods for solving linear systems: Manipulate Eqns; Use Geometry

KEY IDEA:

$\boxed{\text{System of } m \text{ Linear Eqns in } n \text{ variables}} \rightsquigarrow \boxed{\text{Array of coefficients}}$

$$\begin{cases} \underline{\text{Eqn 1}}: & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \underline{\text{Eqn 2}}: & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ & \vdots \\ \underline{\text{Eqn}}_m: & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \rightsquigarrow \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

$$\begin{cases} \text{Eqn 1: } & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \text{Eqn 2: } & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ & \vdots \\ \text{Eqn } m: & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \rightsquigarrow \begin{array}{c} \begin{array}{cccc|c} [x_1] & & & [x_n] & [\text{consts}] \end{array} \\ \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right) \\ \begin{array}{cccc} \text{col}_1 & & & \text{col}_n \end{array} \end{array}$$

$\boxed{m \text{ equations}, \ n \text{ unknowns}} \rightsquigarrow \boxed{m \text{ rows}, \ (n+1) \text{ cols}}$

$a_{11} \cdots a_{1n}$ coefficients of Eqn 1 $\rightsquigarrow$ row 1 of Array

$a_{21}, \ldots, a_{2n}$ ———— Eqn 2 $\rightsquigarrow$ — 2 —

$\vdots$

$a_{m1}, \ldots, a_{mn}$ ———— Eqn n $\rightsquigarrow$ row n —

b_1 constant term for Eqn 1

b_2 ———— Eqn 2

$\vdots$

b_m ———— Eqn m

$\rightsquigarrow$ Last column of the array

(a_{ij} & b_i are fixed) $i = 1, \ldots, m$; $j = 1, \ldots, n$

$$\begin{cases} \text{Eqn 1:} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \text{Eqn 2:} & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ & \vdots \\ \text{Eqn } m: & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \rightsquigarrow \begin{array}{cccc|c} [x_1] & & & [x_n] & [\text{consts}] \\ \left(\begin{matrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{matrix}\right. & \begin{matrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{matrix} & \begin{matrix} \cdots \\ \cdots \\ \\ \cdots \end{matrix} & \begin{matrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{matrix} & \left.\begin{matrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{matrix}\right) \\ \text{col}_1 & & & \text{col}_n & \end{array}$$

$\boxed{m \text{ equations}, \ n \text{ unknowns}} \rightsquigarrow \boxed{m \text{ rows}, \ (n+1) \text{ cols}}$

Def. A matrix of size $m \times n$ is a rectangular array of numbers with m rows & n columns.

Write them as
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = (a_{ij})$$

row index (i), column index (j)

$\boxed{\text{WHY?}}$

- Matrices help us to write down linear system
- ______ are convenient for solving systems

Linear Systems $\rightsquigarrow$ Matrix $B = (A \mid b)$ [Augmented matrix]

- coefficients $\longmapsto$ A $m \times n$ matrix of coefficients (coefficient matrix)
- constant terms $\longmapsto$ b $m \times 1$ —— constants

- LAST TIME: • Manipulated Equations to solve a linear system.
- On the matrix side: manipulate Rows of associated Augmented Matrix

$\boxed{\text{Simpler system} = \text{fewer terms}} \longleftrightarrow \boxed{\text{Simpler Augmented Matrix} = \text{many } 0\text{'s}}$

Elementary Operations for equations / matrices

$\boxed{\text{GOAL:}}$ Find ways to manipulate linear systems <u>without changing the solutions</u>.
("Find an <u>equivalent system</u> that is <u>easier</u> to solve!)

EQUATIONS	MATRICES

EQUATIONS	MATRICES
(1) Interchange 2 Equations $[E_i \leftrightarrow E_j]$	(1) Interchange 2 Rows $[R_i \leftrightarrow R_j]$
(2) Replace an Equation by a <u>non-zero</u> constant multiple α $[E_i \longrightarrow \alpha E_i]$	(2) Replace a Row by a <u>non-zero</u> constant multiple α $[R_i \longrightarrow \alpha R_i]$
(3) Replace an equation by adding to it a constant multiple of a different equation $[E_i \longrightarrow E_i + \alpha E_j]$ $i \neq j$	(3) Replace a Row by adding to it a constant multiple of a different Row $[R_i \longrightarrow R_i + \alpha R_j]$ $i \neq j$

$\boxed{\text{Another example (Row operations)}}$

$$\begin{cases} x_1 + 4x_2 + 3x_3 - x_4 = 0 \\ -x_1 + 2x_2 + 3x_3 - 5x_4 = 6 \\ x_1 + 8x_2 + 7x_3 + 3x_4 = 4 \end{cases}$$

Q: Why use the word "Equivalent"? (implicit symmetry!)

A: Row operations are reversible

(1) Exchange Row_i & Row_j

(2) Multiply Row_i by a non-zero number α

Ex:

$$\left[\begin{array}{cccc|c} 1 & 4 & 3 & -1 & 0 \\ 0 & 6 & 6 & -6 & 6 \\ 0 & 4 & 4 & 4 & 4 \end{array}\right] \xrightarrow[R_2 \to \frac{R_2}{6}]{} \left[\begin{array}{cccc|c} 1 & 4 & 3 & -1 & 0 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 4 & 4 & 4 & 4 \end{array}\right]$$

(3) Add to Row_i a Constant Multiple of Row_j $(j \neq i)$

$$\left[\begin{array}{cccc|c} 1 & 4 & 3 & -1 & 0 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 4 & 4 & 4 & 4 \end{array}\right] \xrightarrow[R_3 \to R_3 + (-4)R_2]{} \left[\begin{array}{cccc|c} 1 & 4 & 3 & -1 & 0 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 8 & 0 \end{array}\right]$$

Conclusion: ① Equivalent Linear Systems $\equiv$ Systems with the same solns

Elementary operations on Equations yield Equivalent Systems

② Define: B & B' are "Row Equivalent" Matrices (write $B \underset{row}{\sim} B'$) if we can transform B into B' via Succesive Elementary Row Op.

③ B $\rightsquigarrow$ Associated linear system = System (B)

$\wr$

B' $\rightsquigarrow$ Associated linear system = System (B')

Then : System (B) is equivalent to the System (B').

$\rightsquigarrow$ $\boxed{\text{ALGORITHM for Solving a Linear System}}$

INPUT: Linear System

OUTPUT: Solution Set ← Need To Know How To write them down.

STEP 1: Write the augmented matrix $B = [A|b]$ of the input system

$\boxed{\text{GAUSS JORDAN ELIMINATION}}$ ← STEP 2 Use elementary Row Operations To go from B To a simpler one = B'

STEP 3 Solve the simpler linear system associated to B' (bottom to Top)

$\boxed{\text{SIMPLER systems} \longleftrightarrow \text{(Reduced) Echelon Matrices}}$

$\boxed{\text{Echelon form}}$ = { • Staircase Shape with 0's <u>below</u> staircase
• Each Step starts with a 1.

$\boxed{\text{Reduced Echelon form}}$ = Echelon form + 0's above each starting 1.