Lecture 2: §1.1 Matrices & Systems of Linear Equations II
§1.2 Echelon forms

Last Time:
- Overview of course
- $\boxed{\text{Main result:}}$ systems of linear equations have either
  1. no solution
  2. a unique solution
  3. infinite many solutions
- Saw 2 methods for solving linear systems → $\boxed{\text{Manipulate Eqns}}$ / → Use Geometry

More on this TODAY!

KEY IDEA:

$\boxed{\text{System of } m \text{ Linear Eqns in } n \text{ variables}}$ ⇝ $\boxed{\text{Array of coefficients}}$

MATRICES!

$$\begin{cases} \text{Eqn 1:} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \text{Eqn 2:} & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ & \vdots \\ \text{Eqn } m\text{:} & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \rightsquigarrow \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

$$\left\{\begin{array}{ll}
\text{Eqn 1:} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
\text{Eqn 2:} & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
\vdots & \\
\text{Eqn } m\text{:} & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{array}\right.
\rightsquigarrow
\begin{array}{c}
\begin{array}{cccc|c} [x_1] & & & [x_n] & [\text{consts}] \end{array} \\
\left(\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{array}\right) \\
\begin{array}{cccc} \text{col}_1 & & & \text{col}_n \end{array}
\end{array}$$

$\boxed{m \text{ equations}, \ n \text{ unknowns}}$ $\rightsquigarrow$ $\boxed{m \text{ rows}, \ (n+1) \text{ cols}}$

$$\left.\begin{array}{lll}
a_{11} \cdots a_{1n} & \text{coefficients of Eqn } 1 \\
a_{21}, \ldots, a_{2n} & \text{———— Eqn } 2 \\
\vdots & \vdots \\
a_{m1}, \ldots, a_{mn} & \text{———— Eqn } n
\end{array}\right\}
\begin{array}{l}
\rightsquigarrow \text{row 1 of Array} \\
\rightsquigarrow \text{—— 2 ——} \\
\\
\rightsquigarrow \text{row } n \text{ ——}
\end{array}$$

$$\left.\begin{array}{ll}
b_1 & \text{constant term for Eqn 1} \\
b_2 & \text{———— Eqn 2} \\
\vdots & \\
b_m & \text{———— Eqn } m
\end{array}\right\}
\rightsquigarrow \text{Last column of the array}$$

(a_{ij} & b_i are fixed) $i = 1, \ldots, m$

$j = 1, \ldots, n$

$$\begin{cases} \text{Eqn 1:} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \text{Eqn 2:} & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ & \vdots \\ \text{Eqn } m\text{:} & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \rightsquigarrow \begin{array}{c} \begin{array}{cccc|c} [x_1] & & & [x_n] & [\text{consts}] \end{array} \\ \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right) \\ \begin{array}{cccc} \text{col}_1 & & & \text{col}_n \end{array} \end{array}$$

$\boxed{m \text{ equations}, n \text{ unknowns}} \rightsquigarrow \boxed{m \text{ rows}, (n+1) \text{ cols}}$

Def. A matrix of size $m \times n$ is a rectangular array of numbers with m rows & n columns.

Write them as $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = (a_{ij})$

row index — i; column index — j

Example $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \end{bmatrix}$ 2×3 matrix $\qquad A = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}$ 2×2 matrix

(square matrix)
#rows = #cols

$\boxed{\text{WHY?}}$

- Matrices help us to write down linear system
- ______ are convenient for solving systems

Linear Systems $\rightsquigarrow$ Matrix $B = (A \mid b)$ [Augmented matrix]

- coefficients $\longmapsto$ A $m \times n$ matrix of coefficients (coefficient matrix)
- constant terms $\longmapsto$ b $m \times 1$ —— constants

<u>Example</u>

$$\begin{cases} x_1 + 3x_2 - 2x_3 = 4 \\ 4x_1 - 5x_3 = 0 \\ -2x_1 + 6x_2 = 24 \end{cases} \xrightarrow{\text{Align variables}} \begin{cases} x_1 + 3x_2 + (-2)x_3 = 4 \\ 4x_1 \qquad\quad + (-5)x_3 = 0 \\ -2x_1 + 6x_2 \qquad\quad = 24 \end{cases}$$

$$\xrightarrow{\text{Add 0's for missing terms}} \begin{cases} x_1 + 3x_2 + (-2)x_3 = 4 \\ 4x_1 + 0x_2 + (-5)x_3 = 0 \\ -2x_1 + 6x_2 + 0x_3 = 24 \end{cases} \rightsquigarrow B = \left[\begin{array}{ccc|c} 1 & 3 & -2 & 4 \\ 4 & 0 & -5 & 0 \\ -2 & 6 & 0 & 24 \end{array}\right]$$

- LAST TIME: • <u>Manipulated Equations</u> To solve a linear system.
- On the matrix side: <u>manipulate Rows</u> of associated Augmented Matrix

$\boxed{\text{Simpler system} = \text{fewer terms}} \longleftrightarrow \boxed{\text{Simpler Augmented Matrix} = \text{many 0's}}$

Elementary Operations for equations / matrices

GOAL: Find ways to manipulate linear systems without changing the solutions.
("Find an equivalent system that is easier to solve!)

EQUATIONS	MATRICES
① Interchange 2 Equations $[E_i \leftrightarrow E_j]$	① Interchange 2 Rows $[R_i \leftrightarrow R_j]$
② Replace an Equation by a non-zero constant multiple α $[E_i \longrightarrow \alpha E_i]$	② Replace a Row by a non-zero constant multiple α $[R_i \longrightarrow \alpha R_i]$
③ Replace an equation by adding to it a constant multiple of a different equation $[E_i \longrightarrow E_i + \alpha E_j]$ $i \neq j$	③ Replace a Row by adding to it a constant multiple of a different Row $[R_i \longrightarrow R_i + \alpha R_j]$ $i \neq j$

EQUATIONS	MATRICES
① Interchange 2 Equations $[E_i \leftrightarrow E_j]$	① Interchange 2 Rows $[R_i \leftrightarrow R_j]$
② Replace an Equation by a non-zero constant multiple α $[E_i \longrightarrow \alpha E_i]$	② Replace a Row by a non-zero constant multiple α $[R_i \longrightarrow \alpha R_i]$
③ Replace an equation by adding to it a constant multiple of a different equation $[E_i \longrightarrow E_i + \alpha E_j]$ $i \neq j$	③ Replace a Row by adding to it a constant multiple of a different Row $[R_i \longrightarrow R_i + \alpha R_j]$ $i \neq j$

Example:

$$\begin{cases} x + y = 3 \\ -x + 3y = 9 \end{cases} \xrightarrow[\boxed{E_2 \to E_2 + E_1}]{} \begin{cases} x + y = 3 \\ 4y = 12 \end{cases} \xrightarrow[\boxed{E_2 \to \frac{1}{4} E_2}]{} \begin{cases} x + y = 3 \\ y = 3 \end{cases} \xrightarrow[\boxed{E_1 \to E_1 - E_2}]{} \begin{cases} x = 0 \\ y = 3 \end{cases}$$

EASIER TO SOLVE

EVER EASIER TO SOLVE!

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ -1 & 3 & 9 \end{array}\right] \xrightarrow[\boxed{R_2 \to R_2 + R_1}]{} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 4 & 12 \end{array}\right] \xrightarrow[\boxed{R_2 \to \frac{1}{4} R_2}]{} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 3 \end{array}\right]$$

ECHELON

REDUCED ECHELON

NOTICE: At every step, we get augmented matrices of the systems above

Another example (Row operations)

$$\begin{cases} x_1 + 4x_2 + 3x_3 - x_4 = 0 \\ -x_1 + 2x_2 + 3x_3 - 5x_4 = 6 \\ x_1 + 8x_2 + 7x_3 + 3x_4 = 4 \end{cases} \rightsquigarrow B = \left[\begin{array}{cccc|c} 1 & 4 & 3 & -1 & 0 \\ -1 & 2 & 3 & -5 & 6 \\ 1 & 8 & 7 & 3 & 4 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} \boxed{1} & 4 & 3 & -1 & 0 \\ -1 & 2 & 3 & -5 & 6 \\ 1 & 8 & 7 & 3 & 4 \end{array}\right] \xrightarrow[R_2 \to R_2 + R_1]{} \left[\begin{array}{cccc|c} 1 & 4 & 3 & -1 & 0 \\ 0 & 6 & 6 & -6 & 6 \\ 1 & 8 & 7 & 3 & 4 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - R_1]{} \left[\begin{array}{cccc|c} 1 & 4 & 3 & -1 & 0 \\ 0 & 6 & 6 & -6 & 6 \\ 0 & 4 & 4 & 4 & 4 \end{array}\right] \xrightarrow[R_2 \to \frac{R_2}{6}]{} \left[\begin{array}{cccc|c} 1 & 4 & 3 & -1 & 0 \\ 0 & \boxed{1} & 1 & -1 & 1 \\ 0 & 4 & 4 & 4 & 4 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 4R_2]{} \left[\begin{array}{cccc|c} 1 & 4 & 3 & -1 & 0 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 8 & 0 \end{array}\right] \xrightarrow[R_3 \to \frac{R_3}{8}]{} \left[\begin{array}{cccc|c} 1 & 4 & 3 & -1 & 0 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

ECHELON

(STAIRCASE)

$$\rightsquigarrow \begin{cases} x_1 + 4x_2 + 3x_3 - x_4 = 0 \\ x_2 + x_3 - x_4 = 1 \\ x_4 = 0 \end{cases}$$

$x_1 + 4x_2 + 3x_3 - x_4 = 0 \rightsquigarrow x_1 + 4(-x_3+1) + 3x_3 = 0$

$x_2 + x_3 - x_4 = 1 \rightsquigarrow x_2 = -x_3 + 1$

$x_1 - x_3 + 4 = 0$

$\boxed{x_1 = 4 + x_3}$

$x_4 = 0 \rightsquigarrow x_4 = 0$

(FEWER TERMS, SOLVE FROM BOTTOM TO TOP!)

Soln: $(x_1, x_2, x_3, x_4) = (4 + x_3, -x_3 + 1, x_3, 0) = (4, -1, 0, 0) + x_3(1, -1, 1, 0)$ x_3 free

Q: Why use the word "Equivalent"? (implicit symmetry!)

A: Row operations are <u>reversible</u>

① Exchange Row_i & Row_j ⇝ Exchange back to reverse it

② Multiply Row_i by a non-zero number α

Ex:

$$\left[\begin{array}{cccc|c} 1 & 4 & 3 & -1 & 0 \\ 0 & 6 & 6 & -6 & 6 \\ 0 & 4 & 4 & 4 & 4 \end{array}\right] \xrightarrow{R_2 \to \frac{R_2}{6}} \left[\begin{array}{cccc|c} 1 & 4 & 3 & -1 & 0 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 4 & 4 & 4 & 4 \end{array}\right]$$

$R_2' \to 6R_2'$

⇝ Reverse by multiplying by $1/\alpha$

③ Add to Row_i a Constant Multiple of Row_j $(j \neq i)$

$$\left[\begin{array}{cccc|c} 1 & 4 & 3 & -1 & 0 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 4 & 4 & 4 & 4 \end{array}\right] \xrightarrow{R_3 \to R_3 + (-4)R_2} \left[\begin{array}{cccc|c} 1 & 4 & 3 & -1 & 0 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 8 & 0 \end{array}\right]$$

$R_3' \to R_3' + 4R_2'$

⇝ Reverse of $R_i \to R_i + \alpha R_j$ is $R_i' \to R_i' - \alpha R_j' = R_i + \alpha R_j - \alpha R_j = R_i$

Conclusion: ① Equivalent Linear Systems $\equiv$ Systems with the same solns

Elementary operations on Equations yield Equivalent Systems

② Define: B & B' are "Row Equivalent" Matrices (write $B \underset{\text{row}}{\sim} B'$)
if we can transform B into B' via Succesive Elementary Row Op.

③ $B \rightsquigarrow$ Associated linear system = System (B)
$\wr$
$B' \rightsquigarrow$ Associated linear system = System (B')

Then : System (B) is equivalent to the System (B').

$\rightsquigarrow$ $\boxed{\text{ALGORITHM for Solving a Linear System}}$

INPUT: Linear System
OUTPUT: Solution Set ← Need To Know How to write them down.

STEP 1: Write the augmented matrix $B = [A \mid b]$ of the input system

GAUSS JORDAN ELIMINATION ← STEP 2 Use elementary Row Operations to go from B to a simpler one $= B'$

STEP 3 Solve the simpler linear system associated to B' (bottom to top)

SIMPLER systems $\longleftrightarrow$ (Reduced) Echelon Matrices

Echelon form = { • Staircase Shape with 0's <u>below</u> staircase
• Each Step starts with a 1.

Reduced Echelon form = Echelon form + 0's above each starting 1.

<u>Examples</u>: (1)

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 3 \end{array}\right] \xrightarrow[R_1 \to R_1 - R_2]{} \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 3 \end{array}\right]$$

ECHELON RED. ECHELON

(2)

$$\left[\begin{array}{cccc|c} 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right] \xrightarrow[R_3 \to \frac{1}{3}R_3]{} \left[\begin{array}{cccc|c} 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_2 \to R_2 - R_3]{} \left[\begin{array}{cccc|c} 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

<u>NOT</u> ECHELON ECHELON

$$\xrightarrow[R_1 \to R_1 - R_3]{} \left[\begin{array}{cccc|c} 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_1 \to R_1 - 2R_2]{} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

✓ RED ECHELON

Echelon → Red Echelon

- Fix each <u>Column</u> from <u>Bottom To Top</u>
- Fix Cols from <u>Right To Left</u>