Lecture 3: §1.2 Echelon forms & Gauss Jordan Elimination

Last Time: $\boxed{\text{System of } m \text{ Linear Eqns in } n \text{ variables}}$ $\rightsquigarrow$ $\boxed{m \times (n+1) \text{ Matrix}}$

$$\begin{cases} \text{Eqn 1:} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \text{Eqn 2:} & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ & \vdots \\ \text{Eqn } m: & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \rightsquigarrow B = \left(\begin{array}{cccc|c} \multicolumn{4}{c|}{A} & \\ a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

$$\begin{array}{ccccc} [x_1] & [x_2] & \cdots & [x_n] & \text{const} \end{array}$$

3 Elementary Operations to simplify systems / put matrices in Echelon form

EQUATIONS	MATRICES
① Interchange 2 Equations $[E_i \leftrightarrow E_j]$ $i \neq j$	① Interchange 2 Rows $[R_i \leftrightarrow R_j]$ $i \neq j$
② Replace an Equation by a non-zero constant multiple α $[E_i \longrightarrow \alpha E_i]$	② Replace a Row by a non-zero constant multiple α $[R_i \longrightarrow \alpha R_i]$
③ Replace an equation by adding to it a constant multiple of a different equation $[E_i \longrightarrow E_i + \alpha E_j]$ $i \neq j$	③ Replace a Row by adding to it a constant multiple of a different Row $[R_i \longrightarrow R_i + \alpha R_j]$ $i \neq j$

- $\boxed{\text{Main result:}}$ Elementary operations preserve solution sets (Equiv Systems, Row equiv Matrices)

Favorite Matrices = (Reduced) Echelon form

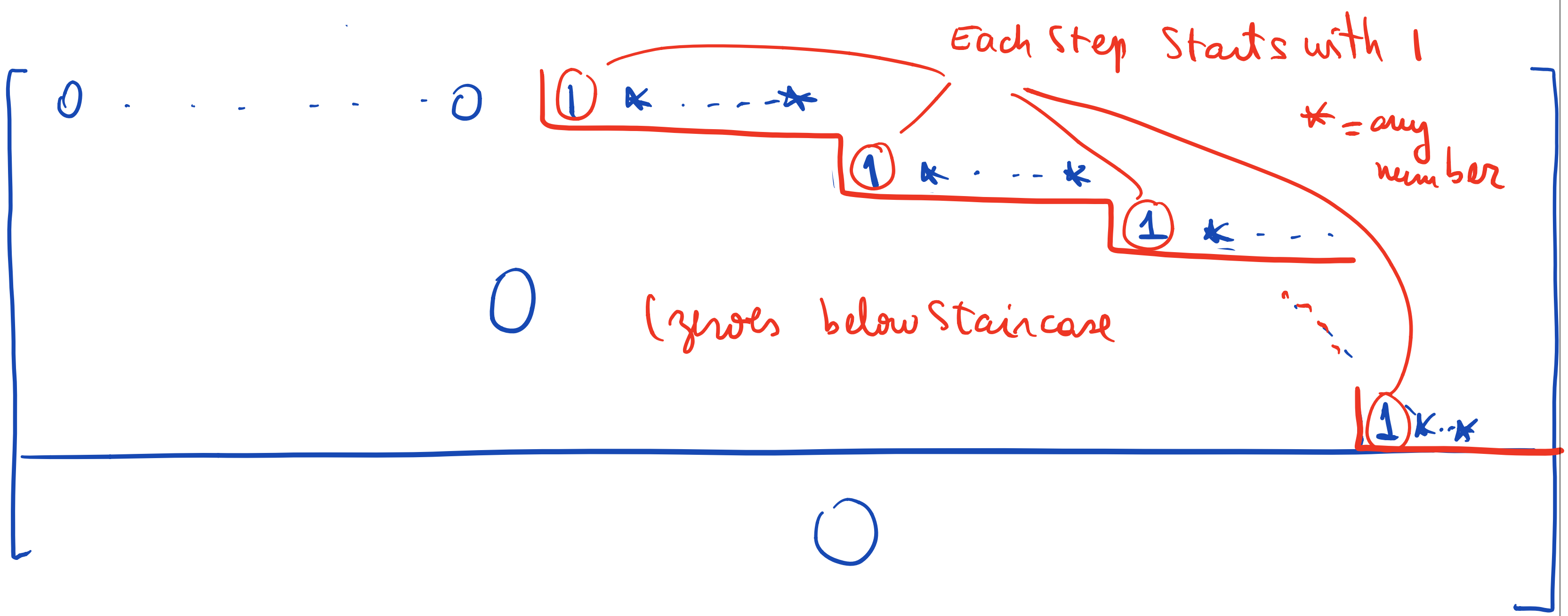
Def: An $m \times n$ matrix A is in <u>echelon form</u> (E.F.) if

(1) all rows containing only 0's are grouped together at the bottom of A

(2) in every nonzero row, the <u>first</u> nonzero entry (from the left) is a 1.

(3) if a row is nonzero, the first nonzero entry is to the <u>right</u> of the first nonzero entry of the previous row ("staircase shape)

<u>In general</u>:



Def An $m \times n$ matrix A is in reduced echelon form (REF) if

(1) A is in EF (staircase shape + start with 1 each nonzero row)

(2) The first nonzero entry in a row is the ONLY nonzero entry in the corresponding column

Q: Why do we care about REF matrices?

A: They give the simplest systems to solve.

$\boxed{\text{Algorithm:}}$ System $\rightsquigarrow B = [A \mid b]$

System $\updownarrow$ Same Solutions! New System

$B = [A \mid b] \xrightarrow{\text{Row operations}} B' = [A' \mid b']$ with A' in REF

- dependent vars
- indep vars

$\boxed{\text{New System easy to solve}} \longleftarrow B' = [A' \mid b']$ with A' in REF

Example 1:

$$\begin{cases} x_2 + x_3 - x_4 = 3 \\ x_1 + 2x_2 - x_3 + x_4 = 1 \\ -x_1 + x_2 + 7x_3 - x_4 = 0 \end{cases} \rightsquigarrow B = \left[\begin{array}{cccc|c} 0 & 1 & 1 & -1 & 3 \\ 1 & 2 & -1 & 1 & 1 \\ -1 & 1 & 7 & -1 & 0 \end{array}\right]$$

Obs: We can check these are solutions of the original system

How? Substitute the values for x_1, x_2, x_3 in the equations!

$$(x_1, x_2, x_3, x_4) = \left(-13-6x_4,\ 17/3+2x_4,\ -8/3-x_4,\ x_4\right)$$

means
$$\begin{cases} x_1 = -13-6x_4 \\ x_2 = \frac{17}{3}+2x_4 \\ x_3 = -\frac{8}{3}-x_4 \end{cases}$$
$\rightsquigarrow$ substitute this on 3 equations & check that we get the expected value

$$\begin{cases} x_2 + x_3 - x_4 = 3 \\ x_1 + 2x_2 - x_3 + x_4 = 1 \\ -x_1 + x_2 + 7x_3 - x_4 = 0 \end{cases}$$

This check does not help us verify we found all the solutions, but we can catch some arithmetic errors this way.

Example 2: $B = \left[\begin{array}{ccc|c} 1 & 0 & 8 & 5 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & 1 \end{array}\right] = B'$

Example 3 $B = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right] = B'$

Theorem: Given a matrix $B = [A|b]$ of size $m \times (n+1)$, there is a unique matrix $B' = [A'|b']$ of size $m \times (n+1)$ that is row equivalent to B' with A' in REF. We find B' using Gauss-Jordan Elimination Algorithm

Obs: We can get A' in many ways, but end result is the same.

Example 4: Solve $\begin{cases} x_2 - x_3 + x_4 - x_5 = 1 \\ x_1 - 3x_2 + x_3 - x_4 + x_5 = 3 \\ -2x_2 + 2x_3 + x_4 - x_5 = 2 \\ x_2 - x_3 + 7x_4 - 7x_5 = 9 \end{cases}$ $\rightsquigarrow B = \left[\begin{array}{ccccc|c} 0 & 1 & -1 & 1 & -1 & 1 \\ 1 & -3 & 1 & -1 & 1 & 3 \\ 0 & -2 & 2 & 1 & -1 & 2 \\ 0 & 1 & -1 & 7 & -7 & 9 \end{array}\right]$

Example 5: Solve $\begin{cases} 2x_1 + 3x_2 - 4x_3 = 3 \\ x_1 - 2x_2 - 2x_3 = -2 \\ -x_1 + 16x_2 + 2x_3 = \frac{33}{2} \end{cases} \rightsquigarrow B = \left[\begin{array}{ccc|c} 2 & 3 & -4 & 3 \\ 1 & -2 & -2 & -2 \\ -1 & 16 & 2 & 33/2 \end{array}\right]$

Obs: Can solve many systems with the same coefficient matrix.

$\left(A \,\middle|\, \underset{\text{system 1}}{b \text{ for}} \,\middle|\, \underset{\text{system 2}}{b \text{ for}} \,\middle|\, \dots \right)$ since operations are always dictated by A

GAUSS-JORDAN ELIMINATION

Input: $B = [A|b]$
m rows, $n+1$ cols

Output: $B' = [A'|b']$ A REF
$B \sim_{\text{row}} B'$.

STEP 0 If B has all zero entries, then $B' = B$

STEP 1 [Find Col] Pick the first (left-most) column with a non-zero entry (call it j)

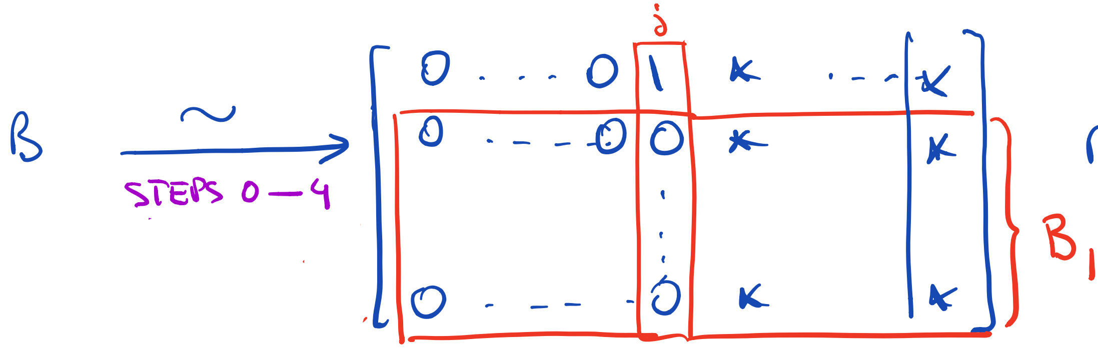
STEP 2 [Swap Step] Exchange Rows so that j^{th} Column has $a_{1j} \neq 0$ (nonzero entry on row 1)

STEP 3 [Rescale Step] If $\alpha = a_{1j}$ is first nonzero entry on row 1, do $R_1 \to \frac{1}{\alpha} R_1$
(After this, new value of $a_{1j} = 1$)

STEP 4 [Replacement Step] Replace each row R_i with $i = 2, 3, \dots m$ with $R_i \to R_i - a_{ij} R_1$

$$\rightsquigarrow \text{get} \quad \begin{array}{c} \begin{array}{ccccccc} & & & j & & & \end{array} \\ \begin{bmatrix} 0 & \cdots & 0 & 1 & * & & * \\ 0 & \cdots & 0 & a_{2j} & * & \cdots & * \\ \vdots & & & \vdots & & & \\ 0 & \cdots & 0 & a_{mj} & * & \cdots & * \end{bmatrix} \end{array} \longrightarrow \begin{array}{c} \begin{array}{ccccccc} & & & \text{Fix} & & & \end{array} \\ \begin{bmatrix} 0 & \cdots & 0 & 1 & * & \cdots & * \\ 0 & \cdots & 0 & 0 & * & & * \\ \vdots & & & \vdots & & & \\ 0 & \cdots & 0 & 0 & * & & * \end{bmatrix} \end{array}$$

B_1



B_1 = m rows & n+1 cols

STEP 5: Repeat Steps 0 through 4 for the smaller matrix B_1 until we get an EF matrix.

STEP 6: [From EF to REF]

Work our way backwards to put 0's on top of each 1 starting a row. (fix columns by moving from right to left)