Lecture 3: §1.2 Echelon forms & Gauss Jordan Elimination

Last Time: $\boxed{\text{System of } m \text{ Linear Eqns in } n \text{ variables}}$ $\rightsquigarrow$ $\boxed{m \times (n+1) \text{ Matrix}}$

$$\begin{cases} \text{Eqn 1: } & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \text{Eqn 2: } & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ & \vdots \\ \text{Eqn } m: & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \rightsquigarrow B = \begin{array}{c} \overbrace{\qquad\qquad\qquad}^{A} \\ \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right) \\ \begin{array}{ccccc} [x_1] & [x_2] & \cdots & [x_n] & \text{cnst} \end{array} \end{array}$$

3 Elementary Operations to simplify systems / put matrices in Echelon form

EQUATIONS	MATRICES
① Interchange 2 Equations $[E_i \leftrightarrow E_j]$ $i \neq j$	① Interchange 2 Rows $[R_i \leftrightarrow R_j]$ $i \neq j$
② Replace an Equation by a non-zero constant multiple α $[E_i \longrightarrow \alpha E_i]$	② Replace a Row by a non-zero constant multiple α $[R_i \longrightarrow \alpha R_i]$
③ Replace an equation by adding to it a constant multiple of a different equation $[E_i \longrightarrow E_i + \alpha E_j]$ $i \neq j$	③ Replace a Row by adding to it a constant multiple of a different Row $[R_i \longrightarrow R_i + \alpha R_j]$ $i \neq j$

• $\boxed{\text{Main result:}}$ Elementary operations preserve solution sets (Equiv Systems, Row equiv Matrices)

Favorite Matrices = (Reduced) Echelon form

Def: An $m \times n$ matrix A is in <u>echelon form</u> (E.F.) if

(1) all rows containing only 0's are grouped together at the bottom of A

(2) in every nonzero row, the <u>first</u> nonzero entry (from the left) is a 1.

(3) if a row is nonzero, the first nonzero entry is to the <u>right</u> of the first nonzero entry of the previous row ("staircase shape")

<u>Example</u>: $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \xrightarrow[R_2 \to \frac{R_2}{2}]{} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ echelon.

not echelon   not echelon

<u>In general</u>:

$$\left[\begin{array}{ccccccccc} 0 & \cdots & 0 & 1 & * & \cdots & * & & \\ & & & & & 1 & * \cdots * & & \\ & & & & & & 1 & * \cdots & \\ & & 0 & & & & & \ddots & \\ & & & & & & & & 1 \; * \cdots * \\ \hline & & & & 0 & & & & \end{array}\right]$$

Each step Starts with 1

$*$ = any number

(zeroes below Staircase

Def An $m \times n$ matrix A is in reduced echelon form (REF) if

(1) A is in EF (staircase shape + start with 1 each nonzero row)

(2) The first nonzero entry in a row is the <u>ONLY</u> nonzero entry on the corresponding column

Ex:

$$\begin{array}{ccc} & & \downarrow \text{issue} \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 4 & 0 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_1 \to R_1 - 4R_2]{} \left[\begin{array}{ccc|c} 1 & 1 & 0 & -6 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

E.F., not REF — REF

Ex: $\begin{cases} x_1 + x_2 = -6 \\ x_3 = 3/2 \\ 0 = 0 \end{cases} \rightsquigarrow \begin{cases} x_1 = -6 - x_2 \\ x_3 = 3/2 \end{cases}$ x_2 any

$(x_1, x_2, x_3) = (-6 - x_2, x_2, 3/2)$
$= (-6, 0, 3/2) + x_2(-1, 1, 0)$

- x_2 indep var
- x_1, x_2 dependent var

Q: Why do we care about REF matrices?

A: They give the simplest systems to solve.

$\boxed{\text{Algorithm:}}$ System $\rightsquigarrow$ $B = [A \mid b]$

$\updownarrow$ Same Solutions! $\qquad \downarrow$ Row operations

$\boxed{\text{New System easy to solve}} \longleftarrow B' = [A' \mid b']$ with A' in REF

- dependent vars
- indep vars

Example 1:

$$\begin{cases} x_2 + x_3 - x_4 = 3 \\ x_1 + 2x_2 - x_3 + x_4 = 1 \\ -x_1 + x_2 + 7x_3 - x_4 = 0 \end{cases} \rightsquigarrow B = \left[\begin{array}{cccc|c} 0 & 1 & 1 & -1 & 3 \\ \boxed{1} & 2 & -1 & 1 & 1 \\ -1 & 1 & 7 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{cccc|c} 0 & 1 & 1 & -1 & 3 \\ 1 & 2 & -1 & 1 & 1 \\ 0 & 3 & 6 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 1 \\ 0 & \boxed{1} & 1 & -1 & 3 \\ 0 & 3 & 6 & 0 & 1 \end{array}\right]$$

FIXED

don't use R_1 until we get EF matrix.

$$\xrightarrow{R_3 \to R_3 - 3R_2} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & 1 & -1 & 3 \\ 0 & 0 & \boxed{3} & 3 & -8 \end{array}\right] \xrightarrow{R_3 \to \frac{R_3}{3}} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & 1 & -1 & 3 \\ 0 & 0 & \boxed{1} & 1 & -8/3 \end{array}\right]$$

FIXED

must become a 1

EF

Use $\boxed{1}$ to make above entries $= 0$.

$$\xrightarrow{R_2 \to R_2 - R_3} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & 0 & -2 & 17/3 \\ 0 & 0 & \boxed{1} & 1 & -8/3 \end{array}\right] \xrightarrow{R_1 \to R_1 + R_3} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 2 & -5/3 \\ 0 & \boxed{1} & 0 & -2 & 17/3 \\ 0 & 0 & 1 & 1 & -8/3 \end{array}\right]$$

x_4

FIXED!

Next: Fix col 2

$$\xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 6 & -29/3 \\ 0 & 1 & 0 & -2 & 17/3 \\ 0 & 0 & 1 & 1 & -8/3 \end{array}\right]$$

$-29/3 = -13$

FIXED

REF

$\rightsquigarrow$ System: $\begin{cases} x_1 + 6x_4 = -13 \\ x_2 - 2x_4 = 17/3 \\ x_3 + x_4 = -8/3 \end{cases}$

So: $x_1 = -13 - 6x_4$, $x_2 = 17/3 + 2x_4$, $x_3 = -8/3 - x_4$

$\rightsquigarrow (x_1, x_2, x_3, x_4) = (-13 - 6x_4, 17/3 + 2x_4, -8/3 - x_4, x_4) = (-13, 17/3, -8/3, 0) + x_4(-6, 2, -1, 1)$

x_4 ANY

So infinitely many solutions!

Obs: We can check these are solutions of the original system

How? Substitute the values for x_1, x_2, x_3 in the equations!

$$(x_1, x_2, x_3, x_4) = \left(-13-6x_4,\ 17/3+2x_4,\ -8/3-x_4,\ x_4\right)$$

means $\begin{cases} x_1 = -13-6x_4 \\ x_2 = \frac{17}{3}+2x_4 \\ x_3 = -\frac{8}{3}-x_4 \end{cases}$ $\rightsquigarrow$ substitute this in 3 equations & check that we get the expected value

$$\begin{cases} x_2 + x_3 - x_4 = 3 & \rightsquigarrow \left(\frac{17}{3}+2x_4\right)+\left(-\frac{8}{3}-x_4\right)-x_4 = \frac{17}{3}-\frac{8}{3}+2x_4-2x_4 = 3 \checkmark \\ x_1 + 2x_2 - x_3 + x_4 = 1 & \rightsquigarrow (-13-6x_4)+2\left(\frac{17}{3}+2x_4\right)-\left(-\frac{8}{3}-x_4\right)+x_4 \\ & \quad = \underbrace{\left(-13+\frac{34}{3}+\frac{8}{3}\right)}_{=1}+\underbrace{(-6+4+1+1)}_{=0}x_4 = 1 \\ -x_1 + x_2 + 7x_3 - x_4 = 0 \end{cases}$$

$$\rightsquigarrow -(-13-6x_4)+\left(\frac{17}{3}+2x_4\right)+7\left(-\frac{8}{3}-x_4\right)-x_4 = \underbrace{\left(13+\frac{17}{3}-\frac{56}{3}\right)}_{=0}+\underbrace{(6+2-7-1)}_{=0}x_4 = 0$$

This check does not help us verify we found all the solutions, but we can catch some arithmetic errors this way.

Example 2: $B = \left[\begin{array}{ccc|c} 1 & 0 & 8 & 5 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & 1 \end{array}\right] = B'$ $\rightsquigarrow$ $\begin{cases} x_1 + 8x_3 = 5 \\ x_2 + 4x_3 = 7 \\ \boxed{0 = 1} \end{cases}$

REF

NO solution!

Example 3 $B = \begin{array}{c} \begin{array}{cccc} & & x_3 & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} = B'$ $\rightsquigarrow$ $\begin{cases} x_1 = 5 \\ x_2 + 4x_3 = 7 \\ 0 = 0 \end{cases}$ $\rightsquigarrow$ $\begin{cases} x_1 = 5 \\ x_2 = 7 - 4x_3 \end{cases}$

REF

Solution: $(x_1, x_2, x_3) = (5, 7-4x_3, x_3) = (5, 7, 0) + x_3(0, -4, 1)$

(parametric form of a line in $\mathbb{R}^3$)

Theorem: Given a matrix $B = [A|b]$ of size $m \times (n+1)$, there is a unique matrix $B' = [A'|b']$ of size $m \times (n+1)$ that is row equivalent to B' with A' in REF. We find B' using Gauss-Jordan Elimination Algorithm

Obs: We can get A' in many ways, but end result is the same.

Example 4: Solve

$$\begin{cases} x_2 - x_3 + x_4 - x_5 = 1 \\ x_1 - 3x_2 + x_3 - x_4 + x_5 = 3 \\ -2x_2 + 2x_3 + x_4 - x_5 = 2 \\ x_2 - x_3 + 7x_4 - 7x_5 = 9 \end{cases} \rightsquigarrow B = \left[\begin{array}{ccccc|c} 0 & 1 & -1 & 1 & -1 & 1 \\ \boxed{1} & -3 & 1 & -1 & 1 & 3 \\ 0 & -2 & 2 & 1 & -1 & 2 \\ 0 & 1 & -1 & 7 & -7 & 9 \end{array}\right]$$

$$\xrightarrow[R_1 \leftrightarrow R_2]{} \left[\begin{array}{ccccc|c} 1 & -3 & 1 & -1 & 1 & 3 \\ 0 & \boxed{1} & -1 & 1 & -1 & 1 \\ 0 & -2 & 2 & 1 & -1 & 2 \\ 0 & 1 & -1 & 7 & -7 & 9 \end{array}\right] \xrightarrow[R_3 \to R_3 + 2R_2]{} \left[\begin{array}{ccccc|c} 1 & -3 & 1 & -1 & 1 & 3 \\ 0 & \boxed{1} & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 3 & -3 & 4 \\ 0 & 1 & -1 & 7 & -7 & 9 \end{array}\right]$$

done with Col 1 ✓

$$\xrightarrow[R_4 \to R_4 - R_2]{} \left[\begin{array}{ccccc|c} 1 & -3 & 1 & -1 & 1 & 3 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & \boxed{3} & -3 & 4 \\ 0 & 0 & 0 & 6 & -6 & 8 \end{array}\right] \xrightarrow[R_3 \to \frac{R_3}{3}]{} \left[\begin{array}{ccccc|c} 1 & -3 & 1 & -1 & 1 & 3 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & \boxed{1} & -1 & 4/3 \\ 0 & 0 & 0 & 6 & -6 & 8 \end{array}\right]$$

done with Cols 1, 2, 3

$$\xrightarrow[R_4 \to R_4 - 6R_3]{} \left[\begin{array}{ccccc|c} 1 & -3 & 1 & -1 & 1 & 3 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 4/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

EF

x_1, x_2, x_4 dependent vars

x_3, x_5 independent vars

Echelon → Red Echelon
• Fix each Column from Bottom To Top • Fix Cols from Right To Left

$$\begin{array}{c} \begin{array}{ccccc} & \downarrow & & \downarrow & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & -3 & 1 & -1 & 1 & 3 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & \boxed{1} & -1 & 4/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \xrightarrow{R_2 \to R_2 - R_3} \left[\begin{array}{ccccc|c} 1 & -3 & 1 & -1 & 1 & 3 \\ 0 & 1 & -1 & 0 & 0 & -1/3 \\ 0 & 0 & 0 & \boxed{1} & -1 & 4/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

EF, not REF

Col 2 & 4 need fixing

$$\xrightarrow{R_1 \to R_1 + R_3} \left[\begin{array}{ccccc|c} 1 & -3 & 1 & 0 & 0 & 13/3 \\ 0 & 1 & -1 & 0 & 0 & -1/3 \\ 0 & 0 & 0 & 1 & -1 & 4/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 + 3R_2} \begin{array}{c} \begin{array}{ccccc} \downarrow & \downarrow & & \downarrow & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & 0 & 10/3 \\ 0 & 1 & -1 & 0 & 0 & -1/3 \\ 0 & 0 & 0 & 1 & -1 & 4/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Col 4 is now fixed!

Col 2 still needs fixing

REF

$\{x_1, x_2, x_3$ dependent (↓)

$\{x_3, x_5$ indep. vars.

Solutions:

$$x_1 - 2x_3 = \frac{10}{3} \rightsquigarrow x_1 = \frac{10}{3} + 2x_3$$

$$x_2 - x_3 = -\frac{1}{3} \rightsquigarrow x_2 = -\frac{1}{3} + x_3$$

$$x_4 - x_5 = 4/3 \rightsquigarrow x_4 = \frac{4}{3} + x_5$$

$$(x_1, x_2, x_3, x_4, x_5) = \left(\frac{10}{3} + 2x_3, -\frac{1}{3} + x_3, x_3, \frac{4}{3} + x_5, x_5\right)$$

$$= \left(\frac{10}{3}, -\frac{1}{3}, 0, \frac{4}{3}, 0\right) + x_3(2, 1, 1, 0, 0) + x_5(0, 0, 0, 1, 1)$$

[x_3, x_5 two free parameters]

Example 5: Solve $\begin{cases} 2x_1+3x_2-4x_3=3 \\ x_1-2x_2-2x_3=-2 \\ -x_1+16x_2+2x_3=\frac{33}{2} \end{cases} \rightsquigarrow B=\left[\begin{array}{ccc|c} \boxed{2} & 3 & -4 & 3 \\ 1 & -2 & -2 & -2 \\ -1 & 16 & 2 & 33/2 \end{array}\right]$

Turn 2 into a 1.

$$\xrightarrow[R_1\to \frac{R_1}{2}]{} \left[\begin{array}{ccc|c} \boxed{1} & 3/2 & -2 & 3/2 \\ 1 & -2 & -2 & -2 \\ -1 & 16 & 2 & 33/2 \end{array}\right] \xrightarrow[\substack{R_2\to R_2-R_1 \\ R_3\to R_3+R_1}]{} \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & \boxed{-7/2} & 0 & -7/2 \\ 0 & 35/2 & 0 & 18 \end{array}\right] \checkmark$$

$$\xrightarrow[R_2\to -\frac{2}{7}R_2]{} \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & 1 & 0 & 1 \\ 0 & 35/2 & 0 & 18 \end{array}\right] \xrightarrow[R_3\to R_3-\frac{35}{2}R_2]{} \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & 1 & 0 & 1 \\ \underbrace{0 \quad\; 0 \quad\; 0}_{EF} & & & 1/2 \end{array}\right]$$

$$\xrightarrow[R_3\to 2R_3]{} \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \begin{cases} x_1+\frac{3}{2}x_2-2x_3=3/2 \\ x_2=1 \\ \boxed{0=1} \end{cases}$$

So **no** solutions!

Obs: Can solve many systems with the same coefficient matrix

$(A \mid b \text{ for system } 1 \mid b \text{ for system } 2 \mid \cdots)$ since operations are always dictated by A

GAUSS-JORDAN ELIMINATION

Input: $B=[A|b]$
m rows, $n+1$ cols

Output: $B'=[A'|b']$ A' REF, $B \sim_{row} B'$.

STEP 0 If B has all zero entries, then $B'=B$

STEP 1 [Find Col] Pick the first (left-most) column with a non-zero entry (call it j)

STEP 2 [Swap Step] Exchange Rows so that j^{th} Column has $a_{1j} \neq 0$ (nonzero entry on row 1)

STEP 3 [Rescale Step] If $\alpha = a_{1j}$ is first nonzero entry on row 1, do $R_1 \rightarrow \frac{1}{\alpha} R_1$
(After this, new value of $a_{1j}=1$)

STEP 4 [Replacement Step] Replace each row R_i with $i=2,3,\dots,m$ with $R_i \rightarrow R_i - a_{ij} R_1$

$$\rightsquigarrow \text{get} \quad \begin{array}{c} \quad\quad\quad\quad\quad j \quad\quad\quad\quad \\ \begin{bmatrix} 0 & \cdots & 0 & 1 & * & \cdots & * \\ 0 & \cdots & 0 & a_{2j} & * & \cdots & * \\ \vdots & & & \vdots & & & \\ 0 & \cdots & 0 & a_{mj} & * & \cdots & * \end{bmatrix} \end{array} \longrightarrow \begin{array}{c} \quad\quad\quad\quad \text{Fix} \quad\quad\quad \\ \begin{bmatrix} 0 & \cdots & 0 & 1 & * & \cdots & * \\ 0 & \cdots & 0 & 0 & * & & * \\ \vdots & & & \vdots & & & \\ 0 & \cdots & 0 & 0 & * & & * \end{bmatrix} \end{array}$$

B_1

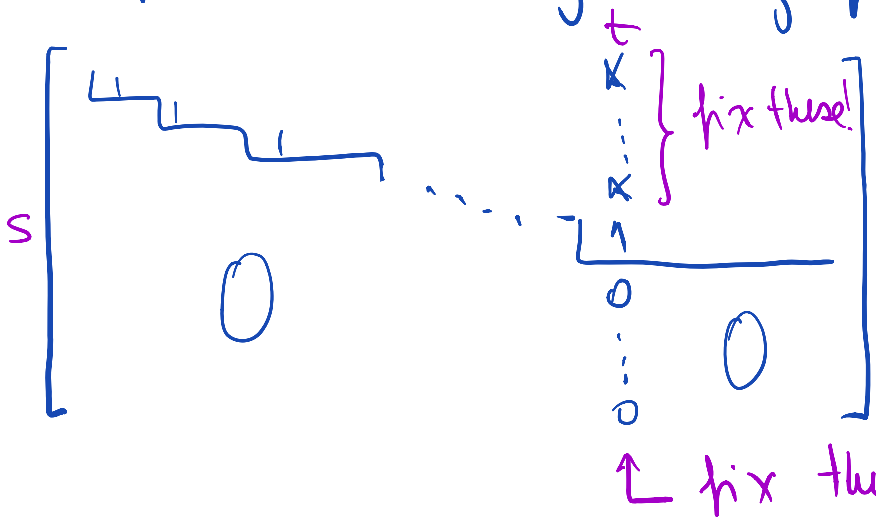
$$B \xrightarrow[\text{STEPS } 0-4]{\sim} \begin{array}{c} \begin{array}{ccccccc} & & & j & & & \end{array} \\ \left[\begin{array}{ccc|c|ccc} 0 & \cdots & 0 & 1 & * & \cdots & * \\ \hline 0 & \cdots & 0 & 0 & * & & * \\ & & & \vdots & & & \\ 0 & \cdots & 0 & 0 & * & & * \end{array}\right] \end{array} \left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\} B_1$$

B_1 = m rows & n+1 cols

STEP 5: Repeat Steps 0 through 4 for the smaller matrix B_1 until we get an EF matrix.

STEP 6: [From EF to REF]

Work our way backwards to put 0's on top of each 1 starting a row. (fix columns by moving from right to left)



$$\xrightarrow{\begin{array}{l} R_1 \to R_1 - a_{1t} R_s \\ R_2 \to R_2 - a_{2t} R_s \\ R_{s-1} \to R_{s-1} - a_{s-1,t} R_{s-1} \end{array}}$$