Lecture 4: §1.3 Consistent Systems of linear equations

Last Time: • Discussed Gauss-Jordan Elimination (Row Reduction) Algorithm to solve linear systems: (Matrix $\rightsquigarrow$ E.F $\rightsquigarrow$ REF)

- $B = [A|b] \xrightarrow[\text{Elementary Row Op.}]{} B' = [A'|b']$ with A' in reduced echelon form (REF)
- **Main Result:** A' is unique; B' is unique if we ask it to be in REF.

Q: Why is this useful? B & B' produce equivalent linear systems (same solns!)

A: It is very easy to solve systems with REF matrices (**Name**: Reduced System)

- Starting 1's give dependent variables
- Rest: independent variables (free parameters)

ALGORITHM:

System with m Eqns, n vars $\longrightarrow$ Augmented Matrix $B=[A|b]$ $\xrightarrow[\text{GAUSS-JORDAN}]{\text{This is the "hard" part}}$ REF matrix $B' \sim B$ $\longrightarrow$ (Reduced) System with m Eqns, n vars $\xrightarrow{\text{EASY part}}$ SOLNS

Example 1: Solve $\begin{cases} 2x_1 + 3x_2 - 4x_3 = 3 \\ x_1 - 2x_2 - 2x_3 = -2 \\ -x_1 + 16x_2 + 2x_3 = 16 \end{cases} \rightsquigarrow B = \left[\begin{array}{ccc|c} 2 & 3 & -4 & 3 \\ 1 & -2 & -2 & -2 \\ -1 & 16 & 2 & 16 \end{array}\right]$

Example 2: Solve $\begin{cases} 2x_1 + 3x_2 - 4x_3 = 3 \\ x_1 - 2x_2 - 2x_3 = -2 \\ \underbrace{-x_1 + 16x_2 + 2x_3}_{\text{same as Ex 1}} = \frac{33}{2} \end{cases} \rightsquigarrow B = \left[\begin{array}{ccc|c} 2 & 3 & -4 & 3 \\ 1 & -2 & -2 & -2 \\ -1 & 16 & 2 & 33/2 \end{array}\right]$

Consistent vs Inconsistent Systems

<u>Def</u>: • A system is <u>Consistent</u> when it admits a solution (either one, or infinitely many)

• If a system has NO solutions, we call it <u>inconsistent</u> or <u>incompatible</u>

<u>Example</u>: 2×2 system $\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$

<u>Example</u> 2×3 system $\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \end{cases}$

Analyzing Consistent Systems

Q1: How many solutions? Do we have any?

Q2: How do we write them down?

$$B = [A \mid b] \underset{\text{row equiv}}{\sim} B' = [A' \mid b'] \text{ in REF}$$

$\boxed{\text{Key fact: We can answer these questions by looking at the matrix } B' !}$

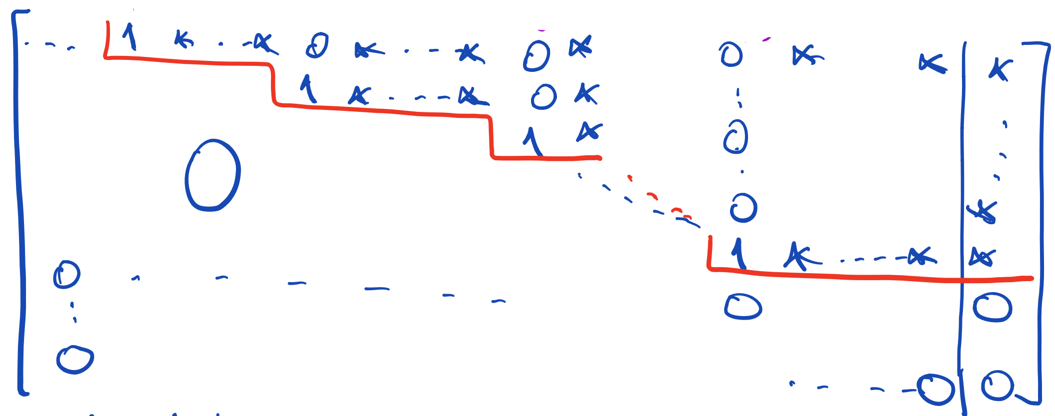
A1:

Example: $B = \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 10 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] = B'$

Q: How to find special solutions?

<u>Summary</u>: System for B has solutions can be checked by looking at B'
(B' cannot have a row $[0 \cdots 0 \; 1]$)

Consistent format for B' =



Last step starts to the left of the bar.

<u>Q2</u>: How do we write down solutions?

$\boxed{\text{Theorem}}$. Assume B has m rows & $n+1$ columns

Write $r = \#$ nonzero rows of B' $= \#$ steps of staircase

Then: Number of dependent variables $= r$

______ independent variables $= n-r$

Q1: How many solutions if system is compatible?

A1: . If we have independent parameters, then infinitely many solns.

. If no independent parameters, then the solution is unique.

Def: $\text{Rank}(B') = r = \#$ nonzero rows of B' $= \#$ dependent param

Note: We are defining this ONLY for REF matrices.

We can extend the definition to any matrix as follows:

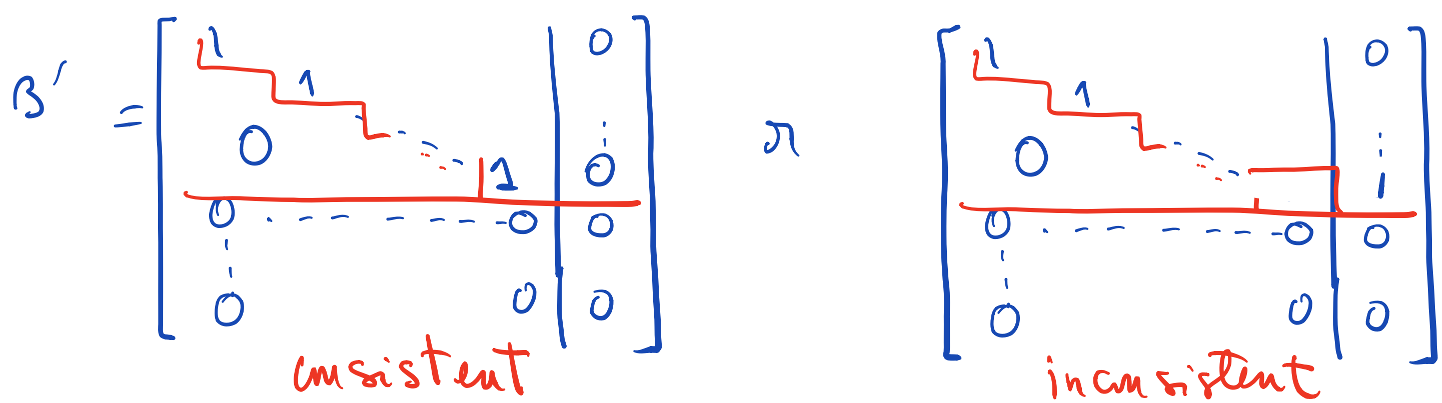
$\text{Rank}(B) = \text{Rank}(B')$ where B' is the unique REF with $B \sim_{\text{row}} B'$.

$$\boxed{\text{Rank}(B') = r = \#\ \text{nonzero rows of } B' = \#\ \text{dependent param}}$$

<u>Consequence</u>

1. $\text{Rank}(B') \leq m \quad (= \#\ \text{rows of } B')$
2. $\text{Rank}(B') \leq n+1 \quad (= \#\ \text{cols of } B')$
3. $\text{Rank}(B') \leq n \quad$ if system is consistent.

<u>Why</u>?



Thm 1: If B' in REF is the augmented matrix of a consistent system of m Eqns in n variables, then

1. $\text{rank}(B') = \#$ nonzero rows of $B' \leq n$
2. there are $n - \text{rank}(B')$ free parameters & we can write our solutions in terms of them.
3. $\text{rank}(B') = n$ if and only if we have a unique soln

Consequence: A system has 3 possibility

1. Inconsistent $\rightsquigarrow$ no solution
2. Consistent & $\text{rank}(B') = n$ $\rightsquigarrow$ unique solution
3. ______ $< n$ $\rightsquigarrow$ infinitely many solns