Lecture 4 : §1.3 Consistent Systems of linear equations

Last Time: • Discussed Gauss-Jordan Elimination (Row Reduction) Algorithm to solve linear systems:

(Matrix $\rightsquigarrow$ E.F $\rightsquigarrow$ REF)

• $B = [A \mid b] \xrightarrow[\text{Row Op.}]{\text{Elementary}} B' = [A' \mid b']$ with A' in reduced echelon form (REF)

• **Main Result**: A' is unique ; B' is unique if we ask it to be in REF.

Q: Why is this useful? B & B' produce equivalent linear systems (same solns!)

A: It is very easy to solve systems with REF matrices (Name: Reduced System)

- Starting 1's give dependent variables
- Rest : independent variables (free parameters)

$\boxed{\text{ALGORITHM:}}$

This is the "hard" part

EASY part

$\boxed{\text{System with } m \text{ Eqns}, n \text{ vars}} \longrightarrow \boxed{\text{Augmented Matrix } B=[A \mid b]} \xrightarrow[\boxed{\text{GAUSS-JORDAN}}]{} \boxed{\text{REF matrix } B' \sim B} \longrightarrow \boxed{\text{(Reduced) System with } m \text{ Eqns}, n \text{ vars}} \longrightarrow \boxed{\text{SOLNS}}$

Example 1: Solve $\begin{cases} 2x_1 + 3x_2 - 4x_3 = 3 \\ x_1 - 2x_2 - 2x_3 = -2 \\ -x_1 + 16x_2 + 2x_3 = 16 \end{cases} \rightsquigarrow B = \left[\begin{array}{ccc|c} \boxed{2} & 3 & -4 & 3 \\ 1 & -2 & -2 & -2 \\ -1 & 16 & 2 & 16 \end{array}\right]$

Turn 2 into a 1.

$$\xrightarrow[R_1 \to \frac{R_1}{2}]{} \left[\begin{array}{ccc|c} \boxed{1} & 3/2 & -2 & 3/2 \\ 1 & -2 & -2 & -2 \\ -1 & 16 & 2 & 16 \end{array}\right] \xrightarrow[\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 + R_1}]{} \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & \boxed{-7/2} & 0 & -7/2 \\ 0 & 35/2 & 0 & 35/2 \end{array}\right] \checkmark$$

$$\xrightarrow[R_2 \to \frac{-2}{7} R_2]{} \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & 1 & 0 & 1 \\ 0 & 35/2 & 0 & 35/2 \end{array}\right] \xrightarrow[R_3 \to R_3 - \frac{35}{2} R_2]{} \underbrace{\left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{EF}$$

fix this!

$$\xrightarrow[R_3 \to 2R_3]{} \underbrace{\left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{EF} \xrightarrow[R_1 \to R_1 - \frac{3}{2} R_2]{} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ REF } \checkmark$$

x_1, x_2 dependent var

x_3 independent var

$$\rightsquigarrow \begin{cases} x_1 \quad - 2x_3 = 0 \\ \quad x_2 \quad = 1 \\ \quad\quad 0 = 0 \end{cases} \quad \begin{array}{l} \rightsquigarrow x_1 = 2x_3 \\ \rightsquigarrow x_2 = 1 \end{array}$$

$$(x_1, x_2, x_3) = (2x_3, 1, x_3)$$

$$= (0, 1, 0) + x_3 (2, 0, 1)$$

Example 2: Solve $\begin{cases} 2x_1 + 3x_2 - 4x_3 = 3 \\ x_1 - 2x_2 - 2x_3 = -2 \\ \underbrace{-x_1 + 16x_2 + 2x_3}_{\text{same as Ex1}} = \frac{33}{2} \end{cases}$ $\rightsquigarrow B = \left[\begin{array}{ccc|c} \boxed{2} & 3 & -4 & 3 \\ 1 & -2 & -2 & -2 \\ -1 & 16 & 2 & 33/2 \end{array}\right]$

Turn 2 into a 1.

$$\xrightarrow[R_1 \to \frac{R_1}{2}]{} \left[\begin{array}{ccc|c} \boxed{1} & 3/2 & -2 & 3/2 \\ 1 & -2 & -2 & -2 \\ -1 & 16 & 2 & 33/2 \end{array}\right] \xrightarrow[\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 + R_1}]{} \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & \boxed{-7/2} & 0 & -7/2 \\ 0 & 35/2 & 0 & 18 \end{array}\right] \checkmark$$

$$\xrightarrow[R_2 \to \frac{-2}{7} R_2]{} \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & 1 & 0 & 1 \\ 0 & 35/2 & 0 & 18 \end{array}\right] \xrightarrow[R_3 \to R_3 - \frac{35}{2} R_2]{} \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & 1 & 0 & 1 \\ \underbrace{0 \quad 0 \quad 0}_{EF} & & & 1/2 \end{array}\right]$$

$$\xrightarrow[R_3 \to 2R_3]{} \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \begin{cases} x_1 + \frac{3}{2}x_2 - 2x_3 = 3/2 \\ x_2 = 1 \\ \boxed{0 = 1} \end{cases}$$

So no solutions!

Obs: Could have solved both systems together (same coefficient matrix)

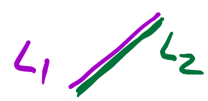
$$\left(A \,\middle|\, \underset{\text{system 1}}{b \text{ for}} \,\middle|\, \underset{\text{system 2}}{b \text{ for}}\right) = \left(\begin{array}{cccc|c|c} 2 & 3 & -4 & 3 & 3 & 3 \\ 1 & -2 & -2 & -2 & -2 & -2 \\ -1 & 16 & 2 & 33/2 & 16 & 33/2 \end{array}\right)$$

Consistent vs Inconsistent Systems

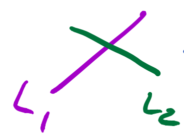
Def:
- A system is <u>Consistent</u> when it admits a solution (either one, or infinitely many)
- If a system has NO solutions, we call it <u>inconsistent</u> or <u>incompatible</u>

Example: 2×2 system $\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases} \longleftrightarrow$ 2 lines in the plane

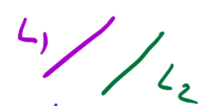
$\left.\begin{array}{l} \text{① If the lines agree} \\ \text{② \_\_\_\_\_\_ are different \& NOT parallel} \end{array}\right.$

① If the lines agree 

infinitely many soln — consistent

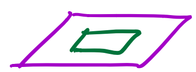
② ______ are different & <u>NOT</u> parallel 

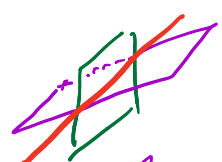
1 soln — consistent

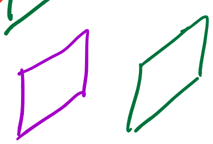
③ ______ but parallel 

no soln $\rightarrow$ inconsistent

Example 2×3 system $\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \end{cases} \longleftrightarrow$ 2 planes in 3-space

① If planes agree 
inf. many solns — consistent

② ______ are different & not parallel 
______ — consistent

③ ______ but parallel 
no soln $\rightarrow$ inconsistent

Analyzing Consistent Systems

Q1: How many solutions? Do we have any?

Q2: How do we write them down?

$$B = [A \mid b] \underset{\text{row equiv}}{\sim} B' = [A' \mid b'] \text{ in REF}$$

Key fact: We can answer these questions by looking at the matrix B'!

A1: Consistent system = no row in B' has the form $[0 \cdots 0 \mid 1]$
(otherwise, we would have eqn $0 = 1$)

Example:

$$B = \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2 \to R_2 - R_3]{R_1 \to R_1 - \frac{3}{2}R_2} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

REF

$\rightsquigarrow$ Inconsistent because last row of B' has the form $[0 \cdots 0 \ 1]$.

Example: $B = \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 10 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] = B'$

Consistent

x_1, x_2, x_4 dependent

x_3 independent

$$\begin{cases} x_1 + 3x_3 = 10 \\ x_2 + 2x_3 = 1 \\ x_4 = 4 \end{cases} \rightsquigarrow \begin{cases} x_1 = 10 - 3x_3 \\ x_2 = 1 - 2x_3 \\ x_4 = 4 \\ x_3 \text{ ANY} \end{cases}$$

Solution

General form of solutions:

$$(x_1, x_2, x_3, x_4) = (10 - 3x_3,\ 1 - 2x_3,\ x_3,\ 4)$$

$$= \underbrace{(10, 1, 0, 4)}_{\text{constant term}} + \underbrace{(-3x_3, -2x_3, x_3, 0)}_{\text{Terms involving } x_3}$$

$$= (10, 1, 0, 4) + x_3(-3, -2, 1, 0)$$

Q: How to find special solutions?

A: Choose values of x_3!

Ex:

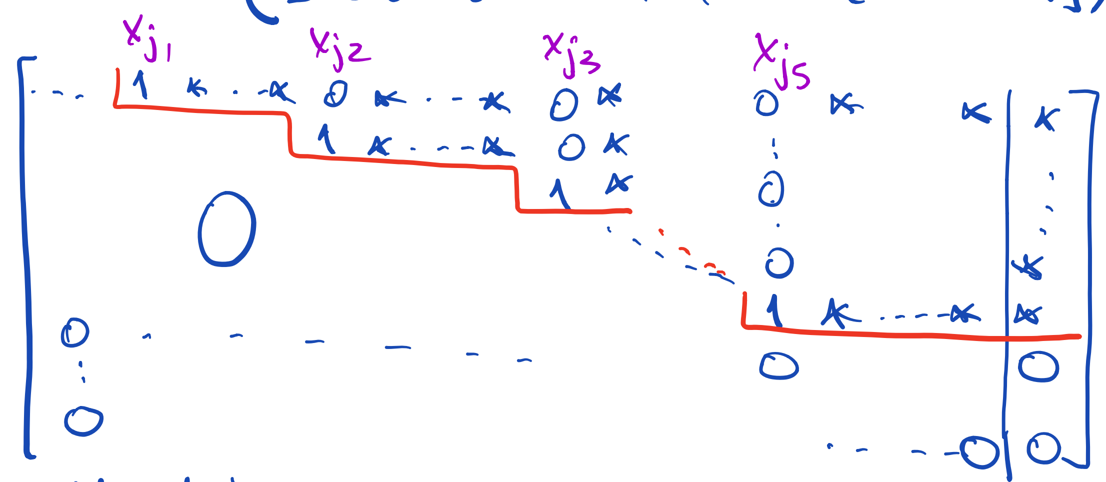
$x_3 = 0$ gives $(10, -1, 0, 4)$

$x_3 = 1$ —— $(7, -3, 1, 4)$

$x_3 = -1$ —— $(13, 1, -1, 4)$

__Summary__: System for B has solutions can be checked by looking at B′ (B′ cannot have a row $[0 \cdots 0 \; 1]$)

Consistent format for B′ =



Last step starts to the left of the bar.

__Q2__: How do we write down solutions?

__A2__ Choose __dependent__ variables = columns where steps in B′ start
($x_{j_1} \cdots\cdots x_{j_s}$) (leading-one variables)

& write these in terms of the remaining variables (called free or independent variables)

Theorem. Assume B has m rows & $n+1$ columns

Write $r = \#$ nonzero rows of B' $= \#$ steps of staircase

Then: Number of dependent variables $= r$

______ independent variables $= n - r$

Q1: How many solutions if system is compatible?

A1:
- If we have independent parameters, then infinitely many solns.
- If no independent parameters, then the solution is unique.

Def: $\text{Rank}(B') = r = \#$ nonzero rows of B' $= \#$ dependent param

Note: We are defining this ONLY for REF matrices.

We can extend the definition to any matrix as follows:

$$\text{Rank}(B) = \text{Rank}(B')$$ where B' is the unique REF with $B \sim_{\text{row}} B'$.

$$\boxed{\text{Rank}(B') = r = \#\text{ nonzero rows of } B' = \#\text{ dependent param}}$$

<u>Consequence</u> ① $\text{Rank}(B') \le m$ $(= \#\text{ rows of } B')$

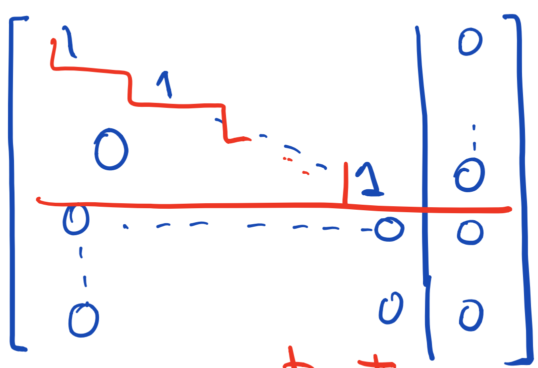
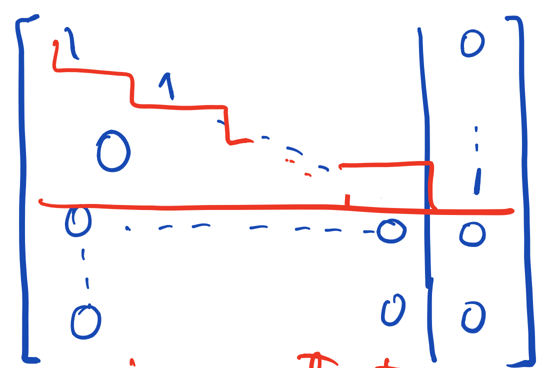
② $\text{Rank}(B') \le n+1$ $(= \#\text{ cols of } B')$

③ $\text{Rank}(B') \le n$ if system is consistent.

Why? ① # nonzero rows $\le$ TOTAL # rows $= m$

② & ③ Each nonzero row starts with a one. As we move downwards, these 1's move to the right, so we can have <u>AT MOST</u> as many steps as the # of cols of B', which is $n+1$.

If B' is consistent, last step starts cannot start at the last col.

$B' =$  or 

consistent

inconsistent

Thm 1: If B' in REF is the augmented matrix of a consistent system of m Eqns in n variables, then

① $\operatorname{rank}(B') = \#$ nonzero rows of $B' \leq n$

② there are $n - \operatorname{rank}(B')$ free parameters & we can write our solutions in terms of them.

③ $\operatorname{rank}(B') = n$ if and only if we have a unique soln

Why? $$B' = \left[\begin{array}{cccc|c} 1 & & & 0 & * \\ & 1 & & & * \\ & & \ddots & & \vdots \\ 0 & & & 1 & * \end{array}\right]$$

$n+1$ cols & n steps before $|$ so each step has length 1 (no room left), so no indep vars.

Consequence: A system has 3 possibility

① Inconsistent $\rightsquigarrow$ no solution

② Consistent & $\operatorname{rank}(B') = n$ $\rightsquigarrow$ unique solution

③ ________________ $< n$ $\rightsquigarrow$ infinitely many solns