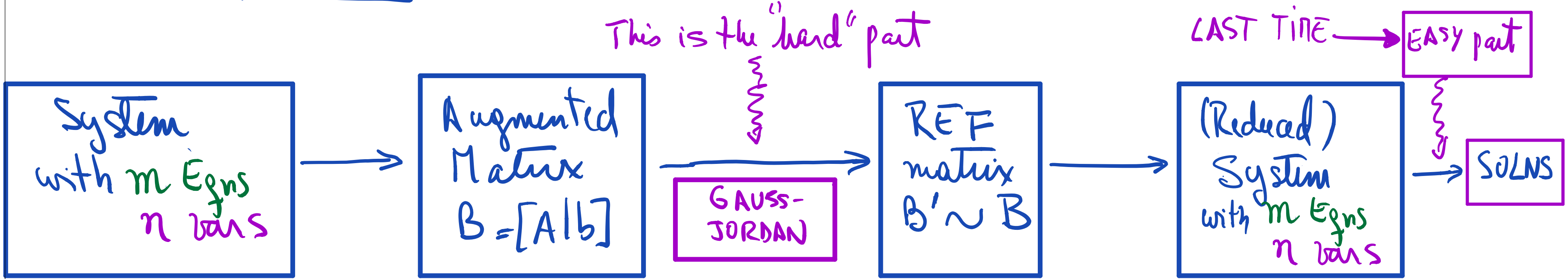
Lecture 5: §1.3 Consistent Systems of linear equations II



Last Time. Consistent Systems = those that have 1 or more solutions

Inconsistent ——— = ——— without solution

- The matrix B' knows what type of system we have!

$$\boxed{\text{Consistent \underline{if and only if}} \qquad\qquad\qquad\qquad\qquad\qquad}$$

- Saw how to write down all solutions.

Writing <u>all</u> solutions of a system

$$B' = \left[\begin{array}{cccc|c} 1 & & & & 0 \\ & 1 & & & \vdots \\ 0 & & \ddots & 1 & 0 \\ \hline 0 & \cdots & \cdots & 0 & 0 \\ \vdots & & & & \\ 0 & & & 0 & 0 \end{array}\right]$$

- 2 types of variables
  1. dependent ones = leading-one variables (coming from columns in B' starting the steps)
  2. independent variables = the rest of the vars.

  This is detected by B'!

- Write solutions to the system using indep. vars as free parameters (our dependent variables will be expressed in terms of indep ones

Consequence 1: A system has either NO solution, a unique one or infinitely many!

$$\boxed{\text{Rank}(B') = r = \#\text{ nonzero rows of } B' = \#\text{ dependent param}}$$

Consequence 2:
1. $\text{Rank}(B') \leq m$ $(= \#\text{ rows of } B')$
2. $\text{Rank}(B') \leq n+1$ $(= \#\text{ cols of } B')$
3. $\text{Rank}(B') \leq n$ if system is consistent

Consequence 3: If compatible system, unique soln if & only if $\text{Rank}(B') = n$

Examples

$\boxed{\text{Rank}(B') = r = \# \text{ nonzero rows of } B' = \# \text{ dependent param}}$

① $B = \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$

② $B = \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Homogeneous vs Inhomogeneous Systems

Def: A system with augmented matrix $B = [A \mid b]$ is homogeneous if all constant terms b_i are zero. Otherwise, we call it inhomogeneous

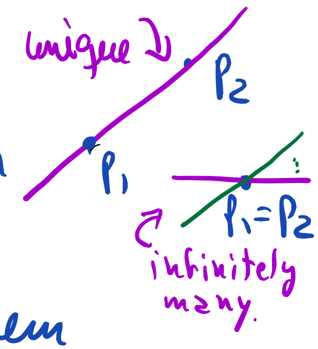
Q: Why do we like homogeneous systems $\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \end{cases}$?

Theorem: A homogeneous $m \times n$ system has infinitely many solutions if $m < n$

Example $B = \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right]$

Geometric Application

- Given 2 pts in the plane, there is always a <u>line</u> through them

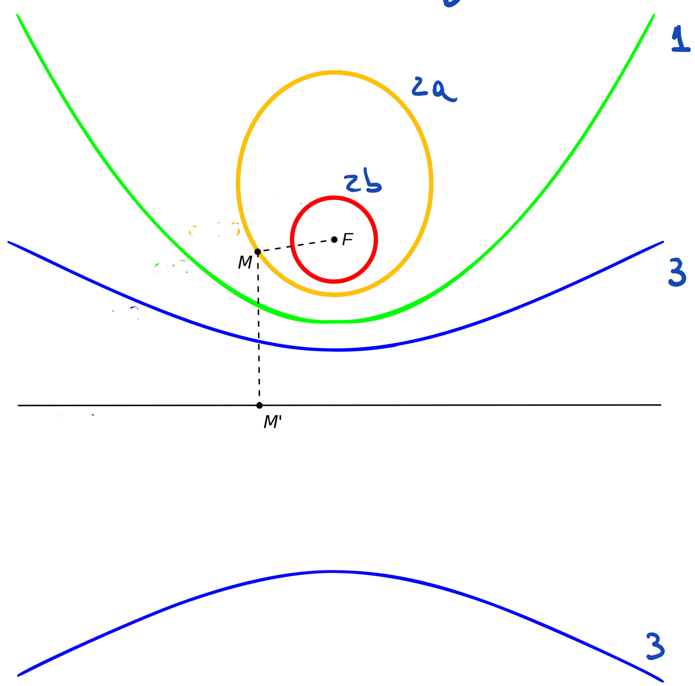


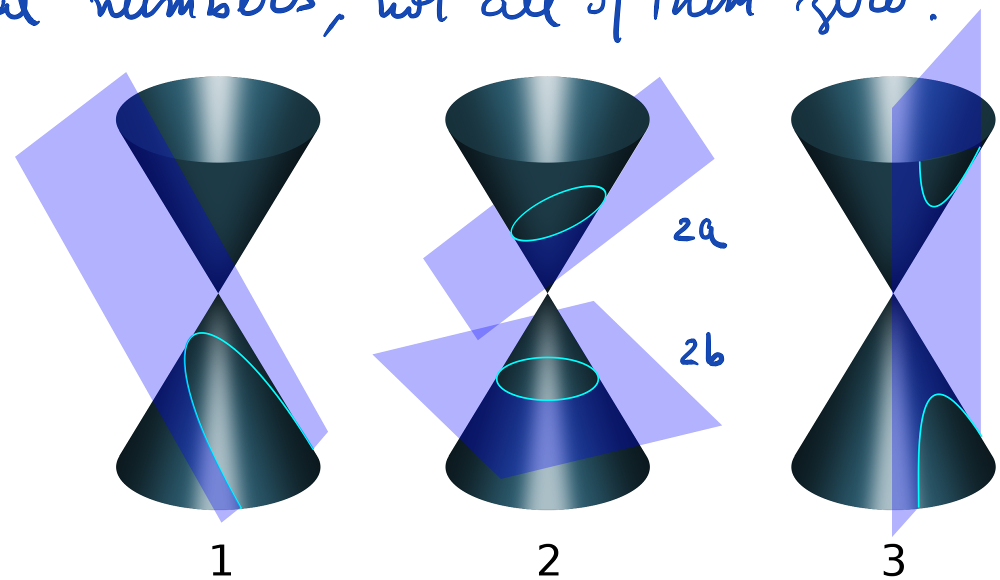
- Given 5 pts in the plane, there is always a <u>conic</u> through them

<u>Def</u>: A <u>conic</u> in the plane is the set of solutions in (x,y) of a degree 2 polynomial

$$\boxed{a x^2 + b xy + c y^2 + d x + f y + g = 0}$$

where a, b, c, d, f, g are fixed real numbers, not all of them zero.





1. parabola

2a ellipse
2b circle
[Conic sections]

3 hyperbola

(Source: wikipedia entry on conic Sections)

(*) $\boxed{a x^2 + b xy + c y^2 + d x + f y + g = 0}$ a, b, c, d, f, g fixed not all zero.

GOAL Given 5 points $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, $P_3 = (x_3, y_3)$, $P_4 = (x_4, y_4)$, $P_5 = (x_5, y_5)$ find the coefficients a, b, c, d, f, g so that $(x_1, y_1), \ldots, (x_5, y_5)$ satisfy (*)

Q: How?

A:

(*) $\boxed{a x^2 + b xy + c y^2 + d x + f y + g = 0}$ a, b, c, d, f, g fixed not all zero.

Example $P_1 = (-1, 0)$, $P_2 = (0, 1)$, $P_3 = (2, 2)$, $P_4 = (2, -1)$ & $P_5 = (0, -3)$