Lecture 5 : §1.3 Consistent Systems of linear equations II

This is the "hard" part | LAST TIME → EASY part

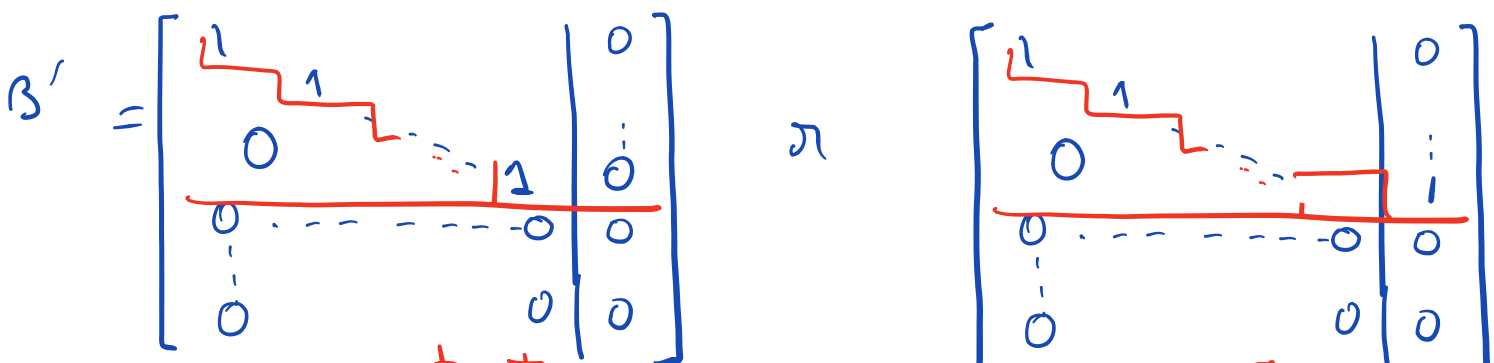
System with m Eqns, n vars → Augmented Matrix $B = [A \mid b]$ $\xrightarrow{\text{GAUSS-JORDAN}}$ REF matrix $B' \sim B$ → (Reduced) System with m Eqns, n vars → SOLNS

Last Time. Consistent Systems = those that have 1 or more solutions

Inconsistent —— = —— without solution

- The matrix B' knows what type of system we have!

$$\boxed{\text{Consistent } \underline{\text{if and only if}} \text{ the matrix } B' \text{ has } \underline{\text{NO}} \text{ row } [0 \cdots 0 \mid 1]}$$



consistent | inconsistent

- Saw how to write down all solutions.

Writing all solutions of a system

$$B' = \left[\begin{array}{cccc|c} 1 & & & & 0 \\ & 1 & & & \vdots \\ 0 & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 0 \\ \vdots & & & & \\ 0 & & & 0 & 0 \end{array}\right]$$

- 2 types of variables
  1. dependent ones = leading-one variables (coming from columns in B' starting the steps)
  2. independent variables = the rest of the vars.

  This is detected by B'!

- Write solutions to the system using indep. vars as free parameters (our dependent variables will be expressed in terms of indep ones

Consequence 1: A system has either NO solution, a unique one or infinitely many!

$$\boxed{\text{Rank}(B') = r = \#\text{ nonzero rows of } B' = \#\text{ dependent params}}$$

Consequence 2:
1. $\text{Rank}(B') \leq m$ (= # rows of B')
2. $\text{Rank}(B') \leq n+1$ (= # cols of B')
3. $\text{Rank}(B') \leq n$ if system is consistent

Consequence 3: If compatible system, unique soln if & only if $\text{Rank}(B') = n$

Examples

$\boxed{\text{Rank}(B') = r = \#\text{ nonzero rows of } B' = \#\text{ dependent param}}$

① $B = \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_1 \to R_1 - \frac{3}{2}R_2 + 2R_3]{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] = B'$ REF

Consistent (no row $[0\,0\,0\,1]$)

$\text{Rank}(B') = 3$, $n = 3$ $\Big\}$ 3 dependent variables (x_1, x_2, x_3), no indep variable $\rightsquigarrow$ $\boxed{\text{Soln (UNIQUE)} \quad x_1 = 0,\ x_2 = 1,\ x_3 = 0}$

② $B = \left[\begin{array}{ccc|c} 1 & 3/2 & -2 & 3/2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_1 \to R_1 - \frac{3}{2}R_2]{} \left[\begin{array}{ccc|c} 1 & 0 & -7/2 & 3/2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = B'$ REF

Consistent (no row $[0\,0\,0\,1]$)

$\text{Rank}(B') = 2$, $n = 3$ $\Big\}$ 2 dependent variables (x_1, x_2), $3-2=1$ indep variable (x_3)

$$\begin{cases} x_1 - \frac{7}{2}x_3 = \frac{3}{2} \\ x_2 + x_3 = 0 \\ 0 \quad 0 \end{cases}$$

$$\rightsquigarrow \begin{cases} x_1 = \frac{3}{2} + \frac{7}{2}x_3 \\ x_2 = -x_3 \\ x_3 \text{ ANY} \end{cases}$$

$$(x_1, x_2, x_3) = \left(\frac{3}{2} + \frac{7}{2}x_3, -x_3, x_3\right) = \left(\frac{3}{2}, 0, 0\right) + x_3\left(\frac{7}{2}, -1, 1\right)$$

- Conclusion 1: We have infinitely many solutions whenever we have free param.
- Conclusion 2: If #Eqns < #Vars, we can NEVER have a unique Soln

Homogeneous vs Inhomogeneous Systems

Def: A system with augmented matrix $B = [A \mid b]$ is homogeneous if all constant terms b_i are zero. Otherwise, we call it inhomogeneous.

Q: Why do we like homogeneous systems
$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \end{cases}$$
?

A: • Gauss-Jordan Elimination preserves the homogeneous property.

• Homogeneous systems are ALWAYS consistent ($x_1 = x_2 = \cdots = x_n = 0$ is always a solution (Name: Trivial or zero solution))

Theorem: A homogeneous $m \times n$ system has infinitely many solutions if $m < n$

Proof: $\text{Rank}(B') \leq m < n$ so we have at least one free parameter. □

Example
$$B = \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right] = B'$$

homogeneous & Consistent (no row $[0\,0\,0\,0\,1]$)

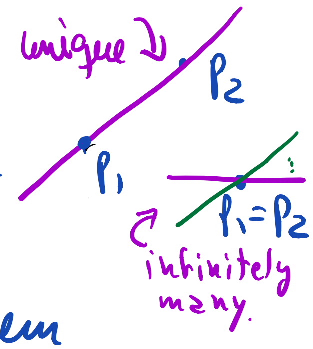
$\left.\begin{array}{l}\text{Rank}(B') = 3 \\ n = 5\end{array}\right\}$ 3 dep vars (x_1, x_2, x_4), 2 indep vars (x_3, x_5)

$$\begin{cases} x_1 - 2x_3 = 0 \\ x_2 - x_3 + x_5 = 0 \\ x_4 - x_5 = 0 \end{cases} \rightsquigarrow \begin{cases} x_1 = 2x_3 \\ x_2 = x_3 + x_5 \\ x_4 = x_5 \\ x_3, x_5 \text{ free} \end{cases}$$

$$\rightsquigarrow (x_1, x_2, x_3, x_4, x_5) = (2x_3, x_3 + x_5, x_3, x_5, x_5)$$
$$= x_3(2,1,1,0,0) + x_5(0,1,0,1,1)$$

Geometric Application

- Given 2 pts in the plane, there is always a line through them

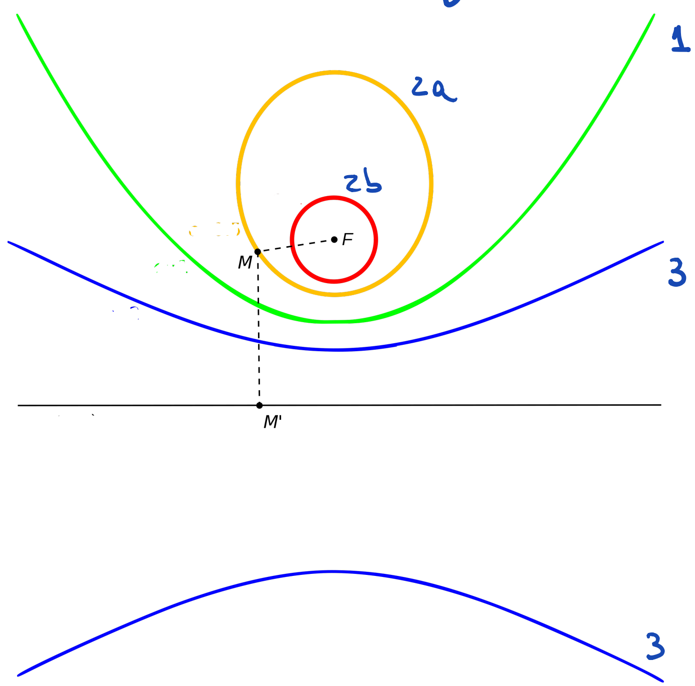


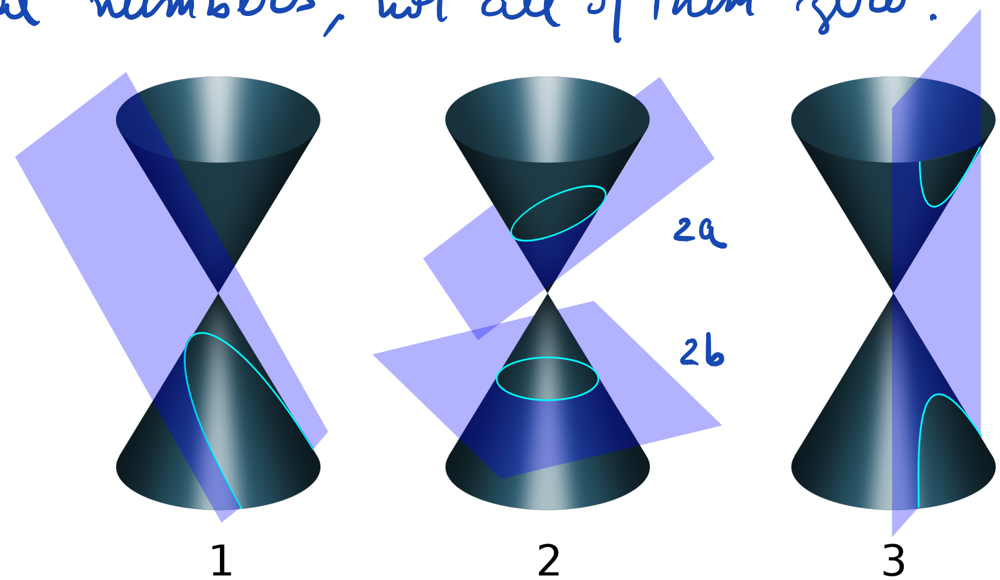
- Given 5 pts in the plane, there is always a conic through them

Def: A conic in the plane is the set of solutions in (x,y) of a degree 2 polynomial

$$\boxed{a x^2 + b xy + c y^2 + d x + f y + g = 0}$$

where a, b, c, d, f, g are fixed real numbers, not all of them zero.





1. parabola

2a ellipse
2b circle

3 hyperbola

(Source: wikipedia entry on conic Sections)

[Conic sections]

(*) $\boxed{a x^2 + b xy + c y^2 + d x + f y + g = 0}$ a, b, c, d, f, g fixed not all zero.

GOAL Given 5 points $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, $P_3 = (x_3, y_3)$, $P_4 = (x_4, y_4)$, $P_5 = (x_5, y_5)$ find the coefficients a, b, c, d, f, g so that $(x_1, y_1), \ldots, (x_5, y_5)$ satisfy (*)

Q: How?

A: Evaluate (*) at each P_i gives a homogeneous linear equation in the variables a, b, c, d, f, g.

So finding the conic through P_1, P_2, P_3, P_4 & P_5 means finding a non trivial solution to the homogeneous system of 5 eqns in $\underbrace{(a, b, c, d, f, g)}_{6 \text{ vars}}$

- Since $5 < 6$ we know we will have infinitely many solns!

- If the points are chosen at random, then $\text{Rank}(B) = 5$, so there will be a unique free parameter, and the choice of (a, b, c, d, f, g) will be unique up to global scalar.

$$(*) \quad \boxed{a x^2 + b xy + c y^2 + d x + f y + g = 0}$$

a, b, c, d, f, g fixed not all zero.

<u>Example</u> $P_1 = (-1, 0)$, $P_2 = (0, 1)$, $P_3 = (2, 2)$, $P_4 = (2, -1)$ & $P_5 = (0, -3)$

$P_1 = (-1, 0) \rightsquigarrow a(-1)^2 + b(-1)\cdot 0 + c \cdot 0^2 + d(-1) + f \cdot 0 + g = 0$

becomes $a - d + g = 0$

The system we get is

$$B = \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{cccccc|c}
a & b & c & d & f & g & \\
\end{array}$$

$$B = \left[\begin{array}{cccccc|c}
1 & -1 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 \\
4 & 4 & 4 & 2 & 2 & 1 & 0 \\
4 & -2 & 1 & 2 & -1 & 1 & 0 \\
0 & 0 & 9 & 0 & -3 & 1 & 0
\end{array}\right]
\begin{array}{l}
P_1 = (-1, 0) \\
P_2\ (0, 1) \\
P_3 = (2, 2) \\
P_4 = (2, -1) \\
P_5 = (0, -3)
\end{array}$$

$$\begin{array}{cccccc}
x^2 & xy & y^2 & x & y & 1
\end{array}$$

$g =$ indep var

$$\sim \left[\begin{array}{cccccc|c}
1 & 0 & 0 & 0 & 0 & 7/18 & 0 \\
0 & 1 & 0 & 0 & 0 & -1/2 & 0 \\
0 & 0 & 1 & 0 & 0 & 1/3 & 0 \\
0 & 0 & 0 & 1 & 0 & -11/18 & 0 \\
0 & 0 & 0 & 0 & 1 & 2/3 & 0
\end{array}\right]$$

$$\rightsquigarrow \begin{aligned}
a &= -\tfrac{7}{18} g \\
b &= \tfrac{1}{2} g \\
c &= -\tfrac{1}{3} g \\
d &= -\tfrac{11}{18} g \\
f &= -\tfrac{2}{3} g
\end{aligned}$$

Take $g = 18$ <u>Conic</u> $-7x^2 + 9xy - 6y^2 + 11x - 12y + 18 = 0$