Lecture 6: §1.5 Matrix Operations

TODAY'S GOALS:

① Define 3 operations on matrices → Addition, Scalar Multiplication } (*) ; Multiplication

② Study the algebra behind these operations (rules that will help us compute faster, like we do with + & × over the reals)

Q: Why do we want to define & study these operations?

A1: We will write linear systems as products of matrices

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases} \rightsquigarrow \underset{A}{\overset{m\times n}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}} \underset{\underline{x}}{\overset{n\times 1}{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}} = \underset{\underline{b}}{\overset{m\times 1}{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}}$$

A2: (*) will be used to define abstract vector spaces (Example: $M_{m\times n}(\mathbb{R}) = \{m\times n \text{ matrices}\}$)

Matrix Operations

Definition A scalar is a real (or complex) number

Definition Two matrices are identical if they have the same size & the same values for all their entries.

In symbols: If $A = [a_{ij}]_{\substack{1\le i\le m \\ 1\le j\le n}}$ & $B = [b_{ij}]_{\substack{1\le i\le k \\ 1\le j\le l}}$, we

$(m\times n)$ matrix $\qquad$ $(k\times l)$ matrix

say $\boxed{A = B}$ if $m=k$ (same # rows) & $a_{ij} = b_{ij}$
$n = l$ (– # cols) for all $i=1,\dots,m$ & for all $j=1,\dots,n$.

- Now that we know what = means for matrices, we can discuss addition (+) & scalar multiplication

Obs: These 2 operations will be used to define (abstract) vector spaces

Matrix Addition & Scalar Multiplication

(a) Given A, B matrices we want to define $A+B$ as a new matrix

<u>Def1</u> If A & B are $m \times n$ matrices, then we define $\boxed{A+B}$ as an $m \times n$ matrix with $[A+B]_{ij}$ = (i,j) entry

(b) Given an $m \times n$ matrix A & a scalar r (real /complex number)

<u>Def 2</u>: The scalar product $r \cdot A$ is an $m \times n$ matrix with

$[r \cdot A]_{ij}$ = (i,j)-entry

Vectors in $\mathbb{R}^{\boxed{n}}$ & solutions to $m \times \boxed{n}$ linear systems

- We identify solutions to linear systems as tuples with n entries.
- Each entry involves constants & multiples of independent variables
- Points in $\mathbb{R}^n$ = ordered tuples with n entries $(x_1, x_2, \ldots, x_n)$
- Column vectors of dimension n = $n \times 1$ matrices $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

- Write $\boxed{\mathbb{R}^n}$ for the set of all column vectors of dimension n

We call $\mathbb{R}^n$ Euclidean Space. It has 2 operations: addition, scalar multiplication

(Think of column vectors as matrices!)

- We can write solutions to linear systems of m Equations in n variables in vector form

Example 2:

$$B = \left[\begin{array}{cccccc|c} 1 & -1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] = B'$$

Consistent?

Conclusion: If an $m \times n$ system with augmented matrix $B = [A|b]$ is consistent, then the general form of a solution is

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$$ if the solution is unique

or

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} +$$ General Solution of the associated homogeneous system with matrix $[A|0]$

$\boxed{\text{Extra Operation on } \mathbb{R}^n = \text{dot product}}$

<u>Def</u>: Given $\underline{u}, \underline{v}$ in $\mathbb{R}^n$, we define their <u>dot product</u> as the number

$$\boxed{\underline{u} \cdot \underline{v}}$$

<u>Def</u>: The <u>norm</u> or <u>magnitude</u> of any $\underline{v}$ in $\mathbb{R}^n$ (Euclidean length) equals

$$\boxed{\|\underline{v}\|} =$$

Matrix Multiplication

Warm-up Case: A matrix of size $m \times n$ & $\underline{x}$ in $\mathbb{R}^n$.

Then $A \cdot \underline{x}$ is in $\mathbb{R}^m$ and it is defined as:

$$\boxed{A \cdot \underline{x}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

In symbols: $(A \cdot \underline{x})_i =$

Application: We can write the system

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

as a product of matrices: $A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$ A = coefficient matrix

General case   How to multiply 2 matrices? $A \cdot B = ???$

ONLY defined when $\boxed{\#\,\text{cols}(A) = \#\,\text{rows}(B)}$

$$\left.\begin{array}{ll} A & m \times s \\ B & s \times n \end{array}\right\} \rightsquigarrow$$

$$\boxed{(AB)_{ij}} =$$

In symbols: