# Lecture 6: §1.5 Matrix Operations

## TODAY'S GOALS:

① Define 3 operations on matrices → Addition, Scalar Multiplication } (*); Multiplication

② Study the algebra behind these operations (rules that will help us compute faster, like we do with + & × over the reals)

**Q:** Why do we want to define & study these operations?

**A1.** We will write linear systems as products of matrices

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases} \rightsquigarrow \underset{A}{\overset{m\times n}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}} \underset{\underline{x}}{\overset{n\times 1}{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}} = \underset{\underline{b}}{\overset{m\times 1}{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}}$$

**A2:** (*) will be used to define abstract vector spaces (Example: $M_{m\times n}(\mathbb{R}) = \{m\times n \text{ matrices}\}$)

# Matrix Operations

<u>Definition</u> A <u>scalar</u> is a real (or complex) number

<u>Definition</u> Two matrices are <u>identical</u> if they have the same size & the same values for all their entries.

In symbols: If $A = [a_{ij}]_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ & $B = [b_{ij}]_{\substack{1 \le i \le k \\ 1 \le j \le l}}$, we
$(m \times n)$ matrix $\qquad$ $(k \times l)$ matrix

say $\boxed{A = B}$ if $m = k$ (same # rows) & $a_{ij} = b_{ij}$
$n = l$ (— # cols) for all $i = 1, \dots, m$ & for all $j = 1, \dots, n$.

<u>Example</u>:

$$\begin{array}{cc} A & B \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq & \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \end{array}$$

because $a_{12} = 0$ but $b_{12} = 1$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

— A has 3 cols but B has 2.

• Now that we know what = means for matrices, we can discuss addition (+) & scalar multiplication

<u>Obs</u>: These 2 operations will be used to define (abstract) vector spaces.

# Matrix Addition & Scalar Multiplication

(a) Given $A$, $B$ matrices we want to define $A+B$ as a new matrix

Def 1 If $A$ & $B$ are $m \times n$ matrices, then we define $\boxed{A+B}$ as an $m \times n$ matrix with $[A+B]_{ij}$ = (i,j) entry $= A_{ij} + B_{ij}$

[ Add entry by entry ]

Ex: ① $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1+3 & 0+1 \\ 0+2 & 1+0 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$

② $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ is not defined (different sizes)

(b) Given an $m \times n$ matrix $A$ & a scalar $r$ ( real / complex number)

Def 2: The scalar product $r \cdot A$ is an $m \times n$ matrix with $[r \cdot A]_{ij}$ = (i,j)-entry $= r \, A_{ij}$ [ multiply all entries by $r$ ]

Ex: $2 \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 & 2 \cdot 0 \\ 2 \cdot 2 & 2 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & -2 \end{bmatrix}$.

# $\boxed{\text{Vectors in } \mathbb{R}^{\boxed{n}} \text{ \& solutions to } m\times \boxed{n} \text{ linear systems}}$

- We identify solutions to linear systems as tuples with $n$ entries.
- Each entry involves constants & multiples of independent variables
- Points in $\mathbb{R}^n$ = ordered tuples with $n$ entries $(x_1, x_2, \ldots, x_n)$
- Column vectors of dimension $n$ = $n\times 1$ matrices $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Ex: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a column vector of dimension 3

- Write $\boxed{\mathbb{R}^n}$ for the set of all column vectors of dimension $n$

We call $\mathbb{R}^n$ Euclidean Space. It has 2 operations: addition, scalar multiplication

(Think of column vectors as matrices!)

- We can write solutions to linear systems of $m$ Equations in $n$ variables in vector form

Example 1 $B = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 3 \end{array}\right] = B'$ $x_1, x_2$ dep, $x_3$ indep.

$$\begin{cases} x_1 - x_3 = 1 \\ x_2 + x_3 = 3 \end{cases} \rightsquigarrow \begin{array}{l} x_1 = 1 + x_3 \\ x_2 = 3 - x_3 \end{array}$$

$$(x_1, x_2, x_3) = (1, 3, 0) + x_3 (1, -1, 1)$$

vs. $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 + 1 \\ -x_3 + 3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

## Example 2:

3×6 system

$$B = \left[\begin{array}{cccccc|c} 1 & -1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] = B'$$

REF

Consistent? YES (no row $[0 \cdots 0 | 1]$)

Cols 1, 4, 6 dependent $(x_1, x_4, x_6)$

Col 2, 3, 5 indep $(x_2, x_3, x_5)$

$$\begin{cases} x_1 - x_2 + 2x_3 = 1 \\ x_4 - x_5 = 2 \\ x_6 = 4 \end{cases} \rightsquigarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1 + x_2 - 2x_3 \\ x_2 \\ x_3 \\ 2 + x_5 \\ x_5 \\ 4 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \\ 4 \end{bmatrix}}_{\substack{\text{constant vector} \\ \text{(particular solution)}}} + \underbrace{x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}}_{\substack{\text{general form of a solution to the} \\ \text{associated HOMOGENEOUS SYSTEM } [A|0]}}$$

Check: $(1\ 0\ 0\ 2\ 0\ 4)$ is a soln

$$\begin{cases} \cdot\ 1 - 0 + 0 = 1 \\ \cdot\ 2 - 0 = 2 \\ \cdot\ 4 = 4 \end{cases} \checkmark$$

Here
$$[A|0] = \left[\begin{array}{cccccc|c} 1 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Conclusion: If an $m \times n$ system with augmented matrix $B = [A|b]$ is consistent, then the general form of a solution is

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$$ if the solution is unique

or

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} +$$ General Solution of the associated homogeneous system with matrix $[A|0]$

Example
$$\begin{cases} x_1 - x_2 + 2x_3 = 0 \\ x_4 - x_5 = 0 \\ x_6 = 0 \end{cases}$$
has general soln
$$\underset{\mathbf{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}} = x_2 \underset{\mathbf{u}}{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}} + x_3 \underset{\mathbf{v}}{\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} + x_5 \underset{\mathbf{w}}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}}$$

Check

| Eqn \ Vector | $\mathbf{u}$ | $\mathbf{v}$ | $\mathbf{w}$ |
|---|---|---|---|
| Eqn 1 | $1 - 1 + 0 = 0$ | $-2 - 0 + 2\cdot 1 = 0$ | $0 - 0 + 2\cdot 0 = 0$ |
| Eqn 2 | $0 - 0 = 0$ | $0 - 0 = 0$ | $1 - 1 = 0$ |
| Eqn 3 | $0 = 0$ | $0 = 0$ | $0 = 0$ |

# Extra Operation on $\mathbb{R}^n$ = dot product

**Def:** Given $\underline{u}, \underline{v}$ in $\mathbb{R}^n$, we define their dot product as the number

$$\boxed{\underline{u} \cdot \underline{v}} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n = \sum_{i=1}^{n} u_i v_i$$

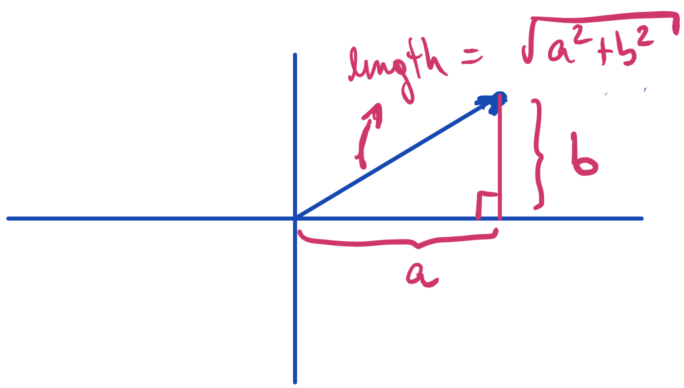
**Example:** $\underline{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\underline{v} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$ $\rightsquigarrow$ $\underline{u} \cdot \underline{v} = 1 \cdot (-1) + 2 \cdot 4 + 3 \cdot 0$

$$= -1 + 8 + 0 = \boxed{7}.$$

**Def:** The norm or magnitude of any $\underline{v}$ in $\mathbb{R}^n$ (Euclidean length) equals

$$\boxed{\|\underline{v}\|} = \sqrt{\underline{v} \cdot \underline{v}} = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

In $\mathbb{R}^2$ $\quad \underline{v} = \begin{bmatrix} a \\ b \end{bmatrix}$




(Phytagoras' Thm)

# Matrix Multiplication

Warm-up Case: $A$ matrix of size $m \times n$ & $\underline{x}$ in $\mathbb{R}^n$.

Then $A \cdot \underline{x}$ is in $\mathbb{R}^m$ and it is defined as:

$$\boxed{A \cdot \underline{x}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

$$= x_1 \, col_1(A) + x_2 \, col_2(A) + \cdots + x_n \, col_n(A)$$

In symbols: $(A \cdot \underline{x})_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = \sum_{j=1}^{n} a_{ij}x_j$

Example $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix}$ $\underline{y} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\rightsquigarrow$ $A \cdot \underline{y} = 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

Application: We can write the system $\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$ as a product of matrices: $A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$ $A$ = coefficient matrix.

Ex: $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is a soln to $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

General case  How to multiply 2 matrices? $A \cdot B = ???$

ONLY defined when $\boxed{\# \text{cols}(A) = \# \text{rows}(B)}$

$$\left.\begin{array}{ll} A & m \times s \\ B & s \times n \end{array}\right\} \rightsquigarrow \quad A \cdot B \text{ is an } m \times n \text{ matrix}$$

and $\text{col}_j(AB) = A \cdot \text{col}_j(B)$ for all $j = 1, \dots, n$.

$$\boxed{(AB)_{ij}} = (A \,\text{col}_j(B))_i = \left( A \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{sj} \end{bmatrix} \right)_i$$

$(i,j)^{th}$ entry $= \boxed{a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{is} b_{sj}} = \sum_{k=1}^{s} a_{ik} b_{kj}$

In symbols: $\text{row}_i(A) \cdot \text{col}_j(B) = (AB)_{ij}$

$$\begin{array}{c} \\ \text{row}_i \\ \\ \end{array} \begin{bmatrix} a_{11} & & a_{1s} \\ \boxed{a_{i1} \; \cdots \; a_{is}} \\ a_{m1} & & a_{ms} \end{bmatrix} \begin{array}{c} \phantom{\text{col}_j} \\ \begin{bmatrix} b_{11} & \cdots & \boxed{\begin{matrix} b_{1j} \\ \vdots \\ b_{sj} \end{matrix}} & \cdots & b_{1n} \\ \vdots & & & & \vdots \\ b_{s1} & & & & b_{sn} \end{bmatrix} \end{array} \rightsquigarrow (AB)_{ij}$$

$\text{col}_j$