Lecture 7: §1.6 Algebraic Properties of Matrix Operations

Last time:

- Defined = on matrices (same size & all entries agree)
- Defined 3 operations on matrices
  - ① Addition (same size matrices)
  - ② Scalar Multiplication
  - ③ Multiplication AB #cols A = #rows B

① A, B $m \times n$ $\rightsquigarrow$ $A+B$ is $m \times n$ also & $[A+B]_{ij} = [A]_{ij} + [B]_{ij}$

② r scalar, A $m \times n$ $\rightsquigarrow$ rA is ——— & $[rA]_{ij} = r[A]_{ij}$

③ Multiplication of matrices: A & B

- $\boxed{\text{Warm-up:}}$ A $m \times n$, $\underline{x}$ $n \times 1$ (col. vector) $\rightsquigarrow$ $A \cdot \underline{x}$ is col vector of dim m!

$$A\,\underline{x} = x_1 \,\mathrm{Col}_1(A) + x_2 \,\mathrm{Col}_2(A) + \cdots + x_n \,\mathrm{Col}_n(A)$$

(Inspired by $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix}$)

- General case: A $m \times s$, B $s \times n$ $\rightsquigarrow$ AB is an $m \times n$ matrix

$$\mathrm{Col}_j(AB) = A\,\mathrm{Col}_j(B) \quad \forall j = 1, \ldots, n$$

$\boxed{\underline{\text{For } A\cdot B}: \ \# \text{ cols}(A) = \# \text{ rows}(B) \ \& \ \text{col}_j(AB) = A\cdot \text{col}_j(B)}$

<u>Examples</u>:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 4 & 1 & 10 \\ 0 & 5 & 1 & 0 \end{bmatrix}$$

1. $\boxed{A\cdot B}$

2. $\boxed{3A}$

Q: Why this definition?

A1 Nice algebraic properties (next time)

A2 Allows for fast substitution (compositions of linear systems

Ex Combine
$$\begin{cases} 1 = 3y_1 - y_2 + y_3 \\ 2 = -3y_1 + 5y_2 \end{cases} \quad \& \quad \begin{cases} y_1 = -4z_1 + z_3 \\ y_2 = z_2 - z_3 \\ y_3 = 0 \end{cases}$$
into a linear system in (z_1, z_2, z_3).

Algebraic Properties

Theorem 1: A, B, C $m \times n$ matrices. Then:

1. [Commutative] $A + B = B + A$
2. [Associative] $(A+B)+C = A+(B+C)$
3. [Neutral Element] The zero matrix $\mathcal{O}$ of size $m \times n$ (all entries are 0) satisfies $A + \mathcal{O} = \mathcal{O} + A = A$ for all $m \times n$ matrices A.
4. [Additive Inverse] Given A, the matrix P of size $m \times n$ with $P_{ij} = -A_{ij}$ solves $A + P = P + A = \mathcal{O}$.

Q: Why is this true?

A:

Obs: $\mathcal{O}$ is sometimes denoted $\mathcal{O}_{m \times n}$ if the size is not clear. $\mathcal{O}_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathcal{O}_{1\times 2} = \begin{bmatrix} 0 & 0 \end{bmatrix}$.

Def: The Identity Matrix of size $n \times n$ (denoted by I_n) is the $n \times n$ matrix with 1's in the diagonal & 0's elsewhere.

$$I_n = \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}}_{n \times n}$$

→ diagonal (i,i) entries

Theorem 2: A of size $m \times n$, B of size $n \times s$, C of size $s \times l$

① [Associative] $\quad \underbrace{(\underset{m\times n}{A}\ \underset{n\times s}{B})}_{m\times s}\underset{s\times l}{C} = \underset{m\times n}{A}\underbrace{(\underset{n\times s}{B}\underset{s\times l}{C})}_{n\times l}$ $\quad m \times l$ matrix.

② [Associative II] $\quad \alpha, \beta$ scalars $\quad \alpha(\beta \underset{m\times n}{A}) = (\alpha\beta)\underset{m\times n}{A}$ $\quad m \times n$

③ [Associative III] $\quad \alpha$ scalar $\quad \alpha(AB) = (\alpha A)B = A(\alpha B)$

④ [Neutral Elements] $\quad A = \underset{m\times m}{I_m}\underset{m\times n}{A} = \underset{m\times n}{A}\underset{n\times n}{I_n}$ $\quad [m \times n]$.

Proof: Explicit Computation of each entry, once sizes have been determined [see Notes / textbook]

Next: Related Addition, multiplication & scalar multiplication.

Theorem 3: ① A, B of size $m \times n$, C of size $n \times l$. Then:

$$(A+B)C = AC + BC$$ [Distribution I]

② A of size $m \times n$, B, C of size $n \times l$. Then:

$$A(B+C) = AB + AC$$ [Distribution II]

③ α, β scalars, A of size $m \times n$. Then:

$$(\alpha+\beta)A = \alpha A + \beta A$$ [Distribution III]

④ α scalar, A & B of size $m \times n$. Then:

$$\alpha(A+B) = \alpha A + \alpha B$$ [Distribution IV]

Transpose of a matrix

IDEA: Transposing means swap the role of rows & columns

Def: Given A of size $m \times n$, the transpose of A is a matrix $\boxed{A^T}$ of size $n \times m$ with

for $i = 1, 2, \ldots, n$
$j = 1, 2, \ldots, m$

Theorem 4: A, B of size $m \times n$, C of size $n \times l$:

① $(A+B)^T = A^T + B^T$

② $(A^T)^T = A$

→ ③ $(AC)^T = C^T A^T$

Def: A is **symmetric** if $\underset{n\times m}{A^T} = \underset{m\times n}{A}$ (in particular $\underline{m=n}$, A is a square matrix)

Proposition: If A has size $m\times n$, then

1. AA^T is symmetric of size
2. A^TA ______________

Proof:

Application: $\underline{v}$ in $\mathbb{R}^n$