Lecture 7: §1.6 Algebraic Properties of Matrix Operations

Last time:

- Defined = on matrices (same size & all entries agree)
- Defined 3 operations on matrices
  - ① Addition (same size matrices)
  - ② Scalar Multiplication
  - ③ Multiplication AB #cols A = #rows B

① A, B $m \times n$ $\rightsquigarrow$ $A+B$ is $m \times n$ also & $[A+B]_{ij} = [A]_{ij} + [B]_{ij}$

② r scalar, A $m \times n$ $\rightsquigarrow$ rA is ——— & $[rA]_{ij} = r[A]_{ij}$

③ Multiplication of matrices: A & B

- $\boxed{\text{Warm-up:}}$ A $m \times n$, $\underline{x}$ $n \times 1$ (col. vector) $\rightsquigarrow$ $A \cdot \underline{x}$ is col vector of dim m!

$$A\,\underline{x} = x_1 \, Col_1(A) + x_2 \, Col_2(A) + \cdots + x_n \, Col_n(A)$$

(Inspired by $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix}$)

- General case: A $m \times s$, B $s \times n$ $\rightsquigarrow$ AB is an $m \times n$ matrix

$$Col_j(AB) = A \, Col_j(B) \quad \forall j = 1, \ldots, n$$

$\boxed{\underline{\text{For } A\cdot B}: \ \#\,\text{cols}(A) = \#\,\text{rows}(B) \ \& \ \text{col}_j(AB) = A\cdot \text{col}_j(B)}$

Examples:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 4 & 1 & 10 \\ 0 & 5 & 1 & 0 \end{bmatrix}$$

2×3 $\qquad$ 3×4

(1) $\boxed{A\cdot B}$ is 2×4

$$AB = \left[A\,\text{col}_1(B) \quad A\,\text{col}_2(B) \quad A\,\text{col}_3(B) \quad A\,\text{col}_4(B) \right] = \begin{bmatrix} -1 & 23 & 6 & 20 \\ 1 & -1 & 0 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = 0\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \text{col}_1(AB)$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 5\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 23 \\ -1 \end{bmatrix} = \text{col}_2(AB)$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1+2+3 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \text{col}_3(AB)$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 10\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 20 \\ 10 \end{bmatrix} = \text{col}_4(AB)$$

(2) $\boxed{BA}$ is not defined because $\left.\begin{array}{l} \#\,\text{cols}(B) = 4 \\ \#\,\text{rows}(A) = 2 \end{array}\right\} \ \& \ 4 \neq 2$

Q: Why this definition?

A1 Nice algebraic properties (next!)

A2 Allows for fast substitution (compositions of linear systems)

Ex Combine $\begin{cases} 1 = 3y_1 - y_2 + y_3 \\ 2 = -3y_1 + 5y_2 \end{cases}$ & $\begin{cases} y_1 = -4z_1 + z_3 \\ y_2 = z_2 - z_3 \\ y_3 = 0 \end{cases}$

into a linear system in (z_1, z_2, z_3)

Use $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -3 & 5 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ & $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -4 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

So $$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -3 & 5 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & -1 & 1 \\ -3 & 5 & 0 \end{bmatrix} \begin{bmatrix} -4 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}}_{= \begin{bmatrix} -12 & -1 & 4 \\ 12 & 5 & -8 \end{bmatrix}} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$\rightsquigarrow \boxed{\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -12 & -1 & 4 \\ 12 & 5 & -8 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}}$$

Algebraic Properties

(Matrix operations are as nice as operations in $\mathbb{R}$.)

Theorem 1: A, B, C $m \times n$ matrices. Then:

1. [Commutative] $A + B = B + A$
2. [Associative] $(A+B)+C = A+(B+C)$
3. [Neutral Element] The zero matrix $\mathbb{O}$ of size $m\times n$ (all entries are 0) satisfies $A + \mathbb{O} = \mathbb{O} + A = A$ for all $m\times n$ matrices A.
4. [Additive Inverse] Given A, the matrix P of size $m\times n$ with $P_{ij} = -A_{ij}$ solves $A + P = P + A = \mathbb{O}$.

Q: Why is this true?

A: Addition is defined entry-by-entry and these properties are true over $\mathbb{R}$ (think of numbers as 1×1 matrices).

Obs: $\mathbb{O}$ is sometimes denoted $\mathbb{O}_{m\times n}$ if the size is not clear

$$\mathbb{O}_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathbb{O}_{1\times 2} = [0\ 0].$$

Def: The <u>Identity Matrix</u> of size $n\times n$ (denoted by I_n) is the $n\times n$ matrix with 1's in the diagonal & 0's elsewhere.

$$I_n = \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}}_{n\times n}$$

diagonal (i,i) entries

Ex: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Theorem 2: A of size $m\times n$, B of size $n\times s$, C of size $s\times l$

1. [Associative] $\underbrace{(\underset{m\times n}{A}\ \underset{n\times s}{B})}_{m\times s}\underset{s\times l}{C} = \underset{m\times n}{A}\underbrace{(\underset{n\times s}{B}\underset{s\times l}{C})}_{n\times l}$ $m\times l$ matrix.

2. [Associative II] α, β scalars $\alpha(\beta \underset{m\times n}{A}) = (\alpha\beta)\underset{m\times n}{A}$ $m\times n$

3. [Associative III] α scalar $\alpha(AB) = (\alpha A)B = A(\alpha B)$

4. [Neutral Elements] $A = \underset{m\times m}{I_m}\ \underset{m\times n}{A} = \underset{m\times n}{A}\ \underset{n\times n}{I_n}$ $[m\times n]$.

Proof: Explicit computation of each entry, once sizes have been determined [see Notes / textbook]

Next: Related Addition, multiplication & scalar multiplication.

Theorem 3: ① A, B of size $m \times n$, C of size $n \times l$. Then:

$$\underbrace{(A+B)}_{m\times n} \underbrace{C}_{n \times l} = \underbrace{\underset{m\times n}{A}\underset{n\times l}{C}}_{m\times l} + \underbrace{BC}_{m\times l}$$

[Distribution I]

$m \times l$ = both sides

② A of size $m \times n$, B, C of size $n \times l$. Then:

$$\underset{m\times n}{A} \underbrace{(B+C)}_{n\times l} = \underbrace{\underset{m\times n}{A}\underset{n\times l}{B}}_{m\times l} + \underbrace{AC}_{m\times l}$$

[Distribution II]

$m \times l$ = both sides

③ α, β scalars, A of size $m \times n$. Then:

$$(\alpha+\beta) A = \underbrace{\alpha A}_{m\times n} + \underbrace{\beta A}_{m\times n}$$

[Distribution III]

$m \times n$ = both sides

④ α scalar, A & B of size $m \times n$. Then:

$$\alpha \left(\underbrace{A+B}_{m\times n}\right) = \underbrace{\alpha A}_{m\times n} + \underbrace{\alpha B}_{m\times n}$$

[Distribution IV]

$m \times n$ = both sides

Transpose of a matrix

IDEA: Transposing means swap the role of rows & columns

Def: Given A of size $m \times n$, the transpose of A is a matrix $\boxed{A^T}$ of size $n \times m$ with $(A^T)_{ij} = A_{ji}$ for $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$

Example

$$A = \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \end{bmatrix}}_{2\times 3} \rightsquigarrow A^T = \underbrace{\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 0 \end{bmatrix}}_{3 \times 2}$$

Theorem 4: A, B of size $m \times n$, C of size $n \times l$:

① $(\underbrace{A+B}_{m\times n})^T = \underbrace{A^T}_{n\times m} + \underbrace{B^T}_{n \times m}$ [$n\times m$ both sides]

② $\underbrace{(\underbrace{A^T}_{n\times m})^T}_{m\times n} = \underbrace{A}_{m\times n}$

③ $(\overbrace{\underbrace{A}_{m\times n} \underbrace{C}_{n\times l}}^{m\times l})^T = \underbrace{C^T}_{l\times n} \underbrace{A^T}_{n\times m}$ [$l \times m$ both sides]

Proof $\boxed{(AC)^T_{ij}} = (AC)_{ji} = A_{j1}C_{1i} + A_{j2}C_{2i} + \dots + A_{jn}C_{ni}$

$$= C_{1i}A_{j1} + C_{2i}A_{j2} + \dots + C_{ni}A_{jn}$$

$$= (C^T)_{i1}(A^T)_{1j} + (C^T)_{i2}(A^T)_{2j} + \dots + (C^T)_{in}(A^T)_{nj} = \boxed{(C^TA^T)_{ij}}$$

Def: A is symmetric if $\underset{n\times m}{A^T} = \underset{m\times n}{A}$ (in particular $m=n$, A is a square matrix)

Proposition: If A has size $m\times n$, then

1. AA^T is symmetric of size $m\times m$
2. A^TA ——— $n\times n$

Proof: • For 1: $(AA^T)^T \underset{\text{Thm 4 (3)}}{=} (A^T)^T A^T \underset{\text{Thm 4 (2)}}{=} AA^T$, so it's symmetric!

• 2 is analogous

Application: $\underline{v}$ in $\mathbb{R}^n$, then $\underline{v}^T$ is a row vector $=$ $1\times n$ matrix

So $\|\underline{v}\| = \sqrt{\underline{v}\cdot\underline{v}} = \sqrt{v_1^2+\dots+v_n^2} = \sqrt{\underline{v}^T\underline{v}}$

$\underline{v}^T = [v_1, \dots v_n]$, $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

Ex: $\underline{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad v^T = [1\ 2]$ $\quad \|v\| = \sqrt{[1\ 2]\begin{bmatrix} 1 \\ 2 \end{bmatrix}} = \sqrt{1^2+2^2} = \sqrt{5}$.