Lecture 8 : §1.9 Matrix Inverses and their properties

Last Time: Studied algebraic properties of +, prod & scalar multiplication for matrices.

① +, x & scalar multiplication interact nicely (Assoc, Distrib, Commutative)

② Additive Neutral Matrix = $\boxed{\mathbb{O} \text{ zero matrix}}$ (solves $A + \mathbb{O} = \mathbb{O} + A = A$ for all A)

③ Additive Inverses exist: Given A an $m \times n$ matrix, the equation in $\underline{\underline{P}}$

$$A + P = P + A = \mathbb{O}$$

has a unique soln: $P = (-1)A$.

④ Multiplicative Neutral Matrix = $\boxed{\text{Identity matrix}}$ $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = \boxed{I_n}$ $n \times n$

$$A = I_m A = A I_n \quad \text{for all } m \times n \text{ matrices } A.$$

⚠ $AB \neq BA$ even if A, B are both $n \times n$ matrices!

⑤ Transpose: swap rows/columns of a matrix
- $(AB)^T = B^T A^T$
- $(A^T)^T = A$

Q What about Multiplicative Inverses? TODAY'S TOPIC!

Inverses of Matrices

Def A matrix A of size $n \times n$ is <u>invertible</u> if we can find a matrix B of size $n \times n$ satisfying

$$\boxed{AB = BA = I_n} \quad (*)$$

Q1 Why square matrices?

A:

Q2: Why do we care?

A:

Proposition: If the multiplicative inverse for A exists, it is unique. Name $= A^{-1}$

Why?

Consequence: If A of size $n \times n$ is invertible, then any system $A \cdot \underline{x} = \underline{b}$

⚠ Not all matrices are invertible!

Q1: How to decide if a matrix is invertible?

Q2: How to build A^{-1} (if it exists)?

TODAY: Build an algorithm that answers both questions!

$$B = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \\ x_{n1} & & x_{nn} \end{bmatrix} \rightsquigarrow \underbrace{A \begin{bmatrix} x_{11} \\ \vdots \\ x_{n1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\text{System 1}}, \quad \underbrace{A \begin{bmatrix} x_{12} \\ \vdots \\ x_{n2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\text{System 2}}, \quad \ldots, \quad \underbrace{A \begin{bmatrix} x_{1n} \\ \vdots \\ x_{nn} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{\text{System } n}$$

- All n systems have the same coefficient matrix, so we solve them ALL Together.

$\boxed{\text{ALGORITHM}}$:

$$\left[\underset{n\times n}{A} \;\middle|\; \underbrace{\begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} \;\middle|\; \begin{matrix} 0 \\ 1 \\ \vdots \\ 0 \end{matrix} \;\middle|\; \cdots \;\middle|\; \begin{matrix} 0 \\ \vdots \\ 1 \end{matrix}}_{=: I_n} \right] \xrightarrow[\text{GAUSS JORDAN}]{} \left[\underset{n\times n}{A'} \;\middle|\; B' \right]$$

with A' in REF

Q: What does A' look like?

Proposition: Fix A $n \times n$ matrix and assume $(A \mid I_n) \sim (I_n \mid B')$, so $AB' = I_n$. Then, $B'A = I_n$ & so $A^{-1} = B'$.

Why?

$\boxed{\text{Theorem: Fix } A \text{ of size } n \times n \text{. Write } (A \mid I_n) \sim (\underbrace{A'}_{\text{REF}} \mid B') \text{. Then } A \text{ is invertible if and only if } A' = I_n \text{. Furthermore, } B' = A^{-1}}$

<u>Example</u>: Take $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Decide if A is invertible. If so, find A^{-1}

Algebraic Properties of Inverses

Theorem: Fix A, C of size $n \times n$, both invertible. Fix $\alpha \neq 0$ scalar

1. A^{-1} is invertible & $(A^{-1})^{-1} = A$.
2. AC ______ & $(AC)^{-1} = C^{-1}A^{-1}$.
3. αA ______ & $(\alpha A)^{-1} = \alpha^{-1} A^{-1} = \frac{1}{\alpha} A^{-1}$.
4. A^T ______ & $(A^T)^{-1} = (A^{-1})^T$.
5. I_n ______ & $(I_n)^{-1} = I_n$.

Proof.

The 2x2 case

Write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ & set $\boxed{\Delta := ad - bc}$

Rule

① If $\Delta = 0$, then A is not invertible.

② If $\Delta \neq 0$, ______ is invertible & $A^{-1} = \frac{1}{\Delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Why?.