Lecture 8: §1.9 Matrix Inverses and their properties

Last Time: Studied algebraic properties of $+$, prod & scalar multiplication for matrices.

1. $+, \times$ & scalar multiplication interact nicely (Assoc, Distrib, Commutative)
2. Additive Neutral Matrix $=$ $\boxed{\mathbb{0} \text{ zero matrix}}$ (solves $A+\mathbb{0} = \mathbb{0}+A = A$ for all A)
3. Additive Inverses exist: Given A an $m \times n$ matrix, the equation in $\underline{\underline{P}}$

$$A + P = P + A = \mathbb{0}$$

has a unique soln: $P = (-1)A$.

4. Multiplicative Neutral Matrix $=$ $\boxed{\text{Identity matrix}}$ $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = \boxed{I_n}$ $n \times n$

$$A = I_m A = A I_n \quad \text{for all } m \times n \text{ matrices } A.$$

⚠ $AB \neq BA$ even if A, B are both $n \times n$ matrices!

5. Transpose: swap rows/columns of a matrix $\begin{cases} \cdot\ (AB)^T = B^T A^T \\ \cdot\ (A^T)^T = A \end{cases}$

Q What about Multiplicative Inverses? TODAY'S TOPIC!

Inverses of Matrices

Def A matrix A of size $n \times n$ is <u>invertible</u> if we can find a matrix B of size $n \times n$ satisfying $\boxed{AB = BA = I_n}$ (*)

Ex:
- I_n is invertible because $I_n I_n = I_n$ (Neutral Element!)
- $\mathbb{O}$ = zero matrix is never invertible because $A\mathbb{O} = \mathbb{O} \neq I_n$.

Q1 Why square matrices?

A: A $m \times n$ $\quad AB = I_n$ then B has size $n \times n$.
But then BA is only defined when $\underset{n}{\#\text{cols } B} = \underset{m}{\#\text{rows } A}$

Q2: Why do we care?

A: If $A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ A $n \times n$ & B solves (*) then:

$$\underset{\overset{\shortparallel}{I_n}}{\boxed{BA}} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \underset{\text{Assoc.}}{=} B\left(A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = B \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

so $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = B \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ solves the system!

Moreover, <u>if inverses are unique</u>, write $B = A^{-1}$. Then, $A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ has a unique soln!

Proposition: If the multiplicative inverse for A exists, it is unique. Name $= A^{-1}$

Why? Imagine B & B' (size $n\times n$) solve the equation, ie $AB = BA = I_n$

$AB' = B'A = I_n$

Then $$\boxed{B} = B I_n \overset{AB'=I_n}{=} B(AB') \overset{\text{Asso}}{=} (BA)B' \overset{BA=I_n}{=} I_n B' = \boxed{B'}$$

Consequence: If A of size $n\times n$ is invertible, then any system $A\cdot \underline{x} = \underline{b}$ has a unique solution, namely $\underline{x} = A^{-1}\underline{b}$.

Ex: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is invertible & $A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ (Check $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$)

So 1 $A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has a unique soln; $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

2 $A\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ———— $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Obs: These 2 solutions give A^{-1}. $A\begin{bmatrix} x_1 & x_3 \\ x_2 & x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has soln $\begin{bmatrix} x_1 & x_3 \\ x_2 & x_4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

Can solve both systems together! $\left(\overbrace{\begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix}}^{A} \;\middle|\; \underbrace{\overset{\text{System 1}}{\begin{matrix} 1 \\ 0 \end{matrix}} \;\middle|\; \overset{\text{System 2}}{\begin{matrix} 0 \\ 1 \end{matrix}}}_{I_2}\right) \longrightarrow \left(\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I_2} \;\middle|\; \overbrace{\begin{matrix} 1 \\ 0 \end{matrix} \;\middle|\; \begin{matrix} -1 \\ 1 \end{matrix}}^{A^{-1}}\right)$

⚠ Not all matrices are invertible!

Example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible

Why? Propose $B = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ solves $AB = I_2$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} \\ 0 & \boxed{0} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \boxed{1} \end{bmatrix}$$ has no solution!

Q1: How to decide if a matrix is invertible? (Later: determinants will be a faster test)

Q2: How to build A^{-1} (if it exists)?

TODAY: Build an algorithm that answers both questions!

How? $\underset{n\times n}{A}\ \underset{n\times n}{B} = \underset{n\times n}{\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}}$ leads to n systems for $B = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \\ x_{n1} & & x_{nn} \end{bmatrix}$ (in n different sets of variables)

$$\underbrace{A\,\mathrm{Col}_1(B) = \mathrm{Col}_1(I_n) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\text{System } 1}, \quad \underbrace{A\,\mathrm{Col}_2(B) = \mathrm{Col}_2(I_n) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\text{System } 2}, \quad \ldots\ldots, \quad \underbrace{A\,\mathrm{Col}_n(B) = \mathrm{Col}_n(I_n) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{\text{System } n}$$

$$B = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \\ x_{n1} & & x_{nn} \end{bmatrix} \rightsquigarrow \underbrace{A\begin{bmatrix} x_{11} \\ \vdots \\ x_{n1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\text{System 1}}, \quad \underbrace{A\begin{bmatrix} x_{12} \\ \vdots \\ x_{n2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\text{System 2}}, \quad \cdots\cdots, \quad \underbrace{A\begin{bmatrix} x_{1n} \\ \vdots \\ x_{nn} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{\text{System } n}$$

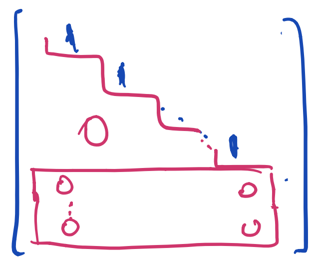
- All n systems have the same coefficient matrix, so we solve them ALL together.

$\boxed{\text{ALGORITHM}}$

$$\Big[\underset{n\times n}{A} \;\Big|\; \underbrace{\begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} \;\Big|\; \begin{matrix} 0 \\ 1 \\ \vdots \\ 0 \end{matrix} \;\Big|\; \cdots \;\Big|\; \begin{matrix} 0 \\ \vdots \\ 1 \end{matrix}}_{=: I_n}\Big] \xrightarrow[\text{GAUSS JORDAN}]{} \Big[\underset{n\times n}{A'} \;\Big|\; B'\Big]$$

with A' in REF

Q: What does A' look like?

$A' =$ 

} may or may not be there

- If $\text{rank}(A') \neq n$, then we cannot have a unique solution for all systems (either inconsistent / infinitely many solns)

But if A' is invertible this cannot happen! So A' is not invertible

- If $\text{rank}(A') = n$, then $A' = I_n$. Each system has a unique soln. (candidate for A^{-1})

Proposition: Fix A $n\times n$ matrix and assume $(A|I_n) \sim (I_n|B')$, so $AB' = I_n$. Then, $B'A = I_n$ & so $A^{-1} = B'$.

Why? Our algorithm gives $(A\,|\,I_n) \underset{\text{GAUSS JORDAN.}}{\overset{(*)}{\sim}} (A'\,|\,B')$

If we want to solve $\boxed{\underset{n\times n}{B'}\,\underset{n\times n}{C} = I_n}$ where C is the unknown matrix

Run the algorithm in reverse! $(B'\,|\,I_n) \overset{\text{reverse}(*)}{\sim} (I_n\,|\,A)$

This means A solves the system! Conclude: $B'A = I_n$.

Theorem: Fix A of size $n\times n$. Write $(A\,|\,I_n) \sim (\underbrace{A'}_{\text{REF}}\,|\,B')$. Then A is invertible if and only if $A' = I_n$. Furthermore, $B' = A^{-1}$

Example: $A = \begin{bmatrix}1&1\\0&1\end{bmatrix} \rightsquigarrow \left[\overbrace{\begin{matrix}1&1\\0&1\end{matrix}}^{A}\,\middle|\,\overbrace{\begin{matrix}1&0\\0&1\end{matrix}}^{I_2}\right] \xrightarrow[R_1\to R_1-R_2]{} \left[\underbrace{\begin{matrix}1&0\\0&1\end{matrix}}_{I_2}\,\middle|\,\underbrace{\begin{matrix}1&-1\\0&1\end{matrix}}_{B'}\right] \rightsquigarrow B' = A^{-1}!$

(Prop: $\left[\underbrace{\begin{matrix}1&-1\\0&1\end{matrix}}_{B'}\,\middle|\,\underbrace{\begin{matrix}1&0\\0&1\end{matrix}}_{I_2}\right] \xrightarrow{R_1\to R_1+R_2} \left[\underbrace{\begin{matrix}1&0\\0&1\end{matrix}}_{I_2}\,\middle|\,\underbrace{\begin{matrix}1&1\\0&1\end{matrix}}_{A}\right] \rightsquigarrow A = (B')^{-1}.$)

Example: Take $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Decide if A is invertible. If so, find A^{-1}

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & 5 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2 \to R_2 - 2R_1]{} \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2 \to -R_2]{} \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - R_1]{} \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1 \to R_1 - R_3]{} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1 \to R_1 - 3R_2]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 3 & 2 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Check: $\underbrace{\begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} -5 & 3 & 2 \\ 2 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{B} = I_3 \quad \rightsquigarrow \quad \boxed{A^{-1} = \begin{bmatrix} -5 & 3 & 2 \\ 2 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}$

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ 2 \\ 0 \end{bmatrix} = -5 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathrm{col}_1(AB),$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \mathrm{col}_2(AB),$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathrm{col}_3(AB).$$

Algebraic Properties of Inverses

Theorem: Fix A, C of size $n \times n$, both invertible. Fix $\alpha \neq 0$ scalar.

1. A^{-1} is invertible & $(A^{-1})^{-1} = A$.
2. AC ——— & $(AC)^{-1} = C^{-1}A^{-1}$. (as with transpose)
3. αA ——— & $(\alpha A)^{-1} = \alpha^{-1}A^{-1} = \frac{1}{\alpha}A^{-1}$.
4. A^T ——— & $(A^T)^{-1} = (A^{-1})^T$.
5. I_n ——— & $(I_n)^{-1} = I_n$.

Proof:
1. $AA^{-1} = A^{-1}A = I_n$ so $A^{-1}B = BA^{-1} = I_n$ has a soln.
2. $(AC)(C^{-1}A^{-1}) = I_n = (C^{-1}A^{-1})(AC)$
3. $\alpha A(\frac{1}{\alpha}A^{-1}) = (\alpha\frac{1}{\alpha})(AA^{-1}) = I_n$; $(\frac{1}{\alpha}A^{-1})(\alpha A) = \frac{1}{\alpha}\alpha(A^{-1}A) = I_n$
4. $A^T(A^{-1})^T = (A^{-1}A)^T = I_n^T = I_n$

   $(A^{-1})^T A^T = (AA^{-1})^T = I_n^T = I_n$
5. $I_n I_n = I_n$.

The 2x2 case

Write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ & set $\boxed{\Delta := ad - bc}$ (determinant)

Rule

1. If $\Delta = 0$, then A is <u>not</u> invertible.
2. If $\Delta \neq 0$, —— is invertible & $A^{-1} = \frac{1}{\Delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

<u>Why</u>? • For 2, we can check both products give I_2.

$$\begin{cases} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{\Delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} ad-bc & -ab+ba \\ cd-dc & -cb+da \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \frac{1}{\Delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} da-bc & db-bd \\ -ca+ac & -cb+ad \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{cases}$$

• For 1, we will try to solve $A\begin{bmatrix} x_1 & x_3 \\ x_2 & x_4 \end{bmatrix} = I_2$ and see we fail if $\Delta = 0$.

Assume $\Delta = ad-bc = 0$ $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} x_1 & x_3 \\ x_2 & x_4 \end{bmatrix}$

We will show either $AB = I_n$ or $BA = I_n$ has no solution.

CASE 1: If $b=0$, then $ad=0$

① If $a=0$, then $\begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}\begin{bmatrix} x_1 & x_3 \\ x_2 & x_4 \end{bmatrix} = \begin{bmatrix} \boxed{0} & 0 \\ * & * \end{bmatrix} = \begin{bmatrix} \boxed{1} & 0 \\ 0 & 1 \end{bmatrix}$ So NO solution!

② If $d=0$, then $\begin{bmatrix} x_1 & x_3 \\ x_2 & x_4 \end{bmatrix}\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} * & 0 \\ * & \boxed{0} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \boxed{1} \end{bmatrix}$

So NO solution!

CASE 2: Assume $b \neq 0$, then $ad=bc$ gives $\boxed{c = \frac{ad}{b}}$

$$[A \mid I_2] = \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ \frac{ad}{b} & d & 0 & 1 \end{array}\right] \xrightarrow[R_2 \to R_2 - \frac{d}{b}R_1]{} \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ \underbrace{0 \quad 0} & & 0 & 1 \end{array}\right]$$

we cannot get I_2 (rank ≤ 1)

Shows that $A\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is inconsistent! $\left[\begin{array}{cc|c} a & b & 0 \\ 0 & 0 & 1 \end{array}\right]$

So A is not invertible!