Lecture 9: §1.7 Linear Independence & Nonsingular Matrices

Last Time:
- Uniqueness of inverses of $n \times n$ matrices (whenever they exist)
- Algorithm to find inverse for A of size $n \times n$

$$\boxed{\text{Compute: } [A \mid I_n] \sim [A' \mid B'] \quad A' \text{ REF}}$$

① If $A' = I_n$, then A is invertible & $B' = A^{-1}$

② If $A' \neq I_n$, then A is not invertible.

- **Prop:** If $A \cdot \underline{x} = \underline{b}$, A of size $n \times n$ is invertible, then the system has a unique solution $\boxed{\underline{x} = A^{-1}\underline{b}}$ for any $\underline{b}$ in $\mathbb{R}^n$.

TODAY:
- Understand better how the Algorithm (*) works (via nonsingular matrices)
- Define & study linear independence of vectors in $\mathbb{R}^n$
- Find other tests for checking when a matrix is invertible.

Nonsingular Matrices

Def: An $n \times n$ matrix A is <u>nonsingular</u> whenever $A\underline{x} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ has a unique solution (the zero solution). Otherwise, we say A is <u>singular</u>

$$\boxed{\text{Prop: } A \text{ is } \underline{\text{nonsingular}} \text{ if, and only if, all } A\underline{x} = \underline{b} \text{ are always consistent, with unique solutions (for } \underline{\text{all}} \text{ choices of constant vectors } \underline{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix})}$$

- This is consistent with our algorithm: pick n values $\underline{b} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$
- If A is nonsingular, then $[A \mid I_n] \sim [I_n \mid \overset{A^{-1}}{\overset{\shortparallel}{B'}}]$, & so A is invertible
- Since $A\underline{x} = \underline{b}$ for A invertible has unique soln $\underline{x} = A^{-1}\underline{b}$, we get:

> **Theorem 1**: A of size $n \times n$ is nonsingular if, and only if, A is invertible

Useful Lemma: Fix P, Q of size $n \times n$, & write $R = PQ$.

If P or Q are singular, then so is R.

Consequence: If $AB = I_n$ with A, B of size $n \times n$, then $BA = I_n$.

Why?

$\boxed{\text{Unit Vectors}}$ In $\mathbb{R}^n$ we have n special vectors of norm 1.

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots\ldots, \quad e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

(Standard Unit Vectors)

Linear Independence in $\mathbb{R}^n$

<u>Def</u>: • A set of vectors $\{\vec{v}_1, \dots, \vec{v}_p\}$ in $\mathbb{R}^n$ is <u>linearly independent</u> (l.i.) if the only solution $(a_1, \dots, a_p)$ to the vector equation:

$$(*) \quad \boxed{a_1 \vec{v}_1 + a_2 \vec{v}_2 + \cdots + a_p \vec{v}_p = \vec{0} \quad \text{in } \mathbb{R}^n}$$

is the trivial one, that is $a_1 = a_2 = \cdots = a_p = 0$.

• If a non-trivial solution to (*) exists, we say the set $\{\vec{v}_1, \dots, \vec{v}_p\}$ is <u>linearly dependent</u> (l.d.)

<u>Example</u>: ① $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is

② $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right]$ is

③ $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$ is

Example: $\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 8 \\ 11 \\ 8 \end{bmatrix}$, $\vec{v}_4 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

1. $\boxed{\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \text{ is l.d.}}$

2. $\boxed{\{\vec{v}_1, \vec{v}_2, \vec{v}_4\} \text{ is l.i.}}$

Q Why study independence?

Proposition 1: The system $\underset{n\times p}{A} \cdot \underline{x} = \underline{0}$ has a unique solution if and only if the p columns of A are l.i. vectors in $\mathbb{R}^n$.

Proposition 2: A of size $n\times p$, $\underline{b}$ in $\mathbb{R}^n$. Then, the system $A\,\underline{x} = \underline{b}$ has a solution if and only if $\underline{b}$ is a linear combination of the p columns of A $\left(x_1 \, Col_1(A) + x_2 \, Col_2(A) + \cdots + x_p \, Col_p(A) = \underline{b} \right)$

The solution is unique only when the columns of A are l.i.

Theorem: If $\{\vec{v}_1, \ldots, \vec{v}_p\}$ are l.d with $a_1\vec{v}_1 + \cdots + a_p\vec{v}_p = \underline{0}$ & $a_i \neq 0$, then $\vec{v}_i$ is a linear comb. of the remaining vectors.

Useful Properties

Prop 0: Standard unit Vectors are li

Prop 1: Reordering vectors preserves linear independency / dep.

Prop 2: A subset of a li set is li

Theorem 3: If $\{\vec{v}_1, \ldots \vec{v}_p\}$ in $\mathbb{R}^n$ & $p > n$, then $\{\vec{v}_1, \ldots \vec{v}_p\}$ is ld.

Theorem 4: A matrix is non-singular if and only if cols are li

Summary of Results

Fix A of size $n \times n$. The following statements are equivalent:

1. A is nonsingular ($A\underline{x} = \underline{0}$ has a unique solution)
2. The n columns of A are l.i.
3. $A\underline{x} = \underline{b}$ has a unique solution for all $\underline{b}$ in $\mathbb{R}^n$.
4. A is invertible
5. A is row equivalent to I_n