Lecture 9: §1.7 Linear Independence & Nonsingular Matrices

Last Time: • Uniqueness of inverses of $n \times n$ matrices (whenever they exist)

• Algorithm to find inverse for A of size $n \times n$

$$\boxed{\text{Compute: } [A \mid I_n] \sim [A' \mid B'] \quad A' \text{ REF}}$$

① If $A' = I_n$, then A is invertible & $B' = A^{-1}$

② If $A' \neq I_n$, then A is not invertible.

• **Prop:** If $A \cdot \underline{x} = \underline{b}$ A of size $n \times n$ is invertible, then the system has a unique solution $\boxed{\underline{x} = A^{-1}\underline{b}}$ for any $\underline{b}$ in $\mathbb{R}^n$.

TODAY: • Understand better how the Algorithm (*) works (via nonsingular matrices)

• Define & study linear independence of vectors in $\mathbb{R}^n$

• Find other tests for checking when a matrix is invertible.

$\boxed{\text{Nonsingular Matrices}}$

<u>Def</u>: An $n\times n$ matrix A is <u>nonsingular</u> whenever $A\underline{x} = \overbrace{\begin{bmatrix}0\\ \vdots \\ 0\end{bmatrix}}^{=\underline{0}}$ has a unique solution (the zero solution). Otherwise, we say A is <u>singular</u>

<u>Example</u> $I_3 = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ is nonsingular ; $A = \begin{bmatrix}1&0\\0&0\end{bmatrix}$ is singular (<u>Solns</u>: $x_2\begin{bmatrix}0\\1\end{bmatrix}$)

> <u>Prop</u>: A is <u>nonsingular</u> if, and only if, all $A\underline{x}=\underline{b}$ are always consistent, with unique solutions (for <u>all</u> choices of constant vectors $\underline{b}=\begin{bmatrix}b_1\\ \vdots\\ b_n\end{bmatrix}$)

<u>Why?</u> • Assume all $A\underline{x}=\underline{b}$ have unique solutions, in particular for $\underline{b}=\underline{0}$. So, A is nonsingular by definition.

• For the converse, assume $A\cdot\underline{x}=\underline{0}$ has a unique solution. This means that $[A|0] \sim [A'|0]$ with A' REF has $\operatorname{rank}(A')=n$ (no indep vars!)

But A' has size $n\times n$, so $A'=I_n$.

The <u>same elementary row operations</u> give $[A|\underline{b}] \sim [I_n|b']$, so $\begin{cases}x_1=b_1'\\ x_2=b_2'\\ \vdots\\ x_n=b_n'\end{cases}$ is the unique solution (of course b' will change with $\underline{b}$).

- This is consistent with our algorithm: pick n values $\underline{b} = \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}, \ldots, \begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}$

- If A is nonsingular, then $[A \mid I_n] \sim [I_n \mid \underset{A^{-1}}{\underset{\shortparallel}{B'}}]$, & so A is invertible

- Since $A\underline{x} = \underline{b}$ for A invertible has unique soln $\underline{x} = A^{-1}\underline{b}$, we get:

> **Theorem 1**: A of size $n\times n$ is nonsingular if, and only if, A is invertible

Useful Lemma: Fix P, Q of size $n \times n$, & write $R = PQ$.

If P or Q are singular, then so is R.

Why? (1) If Q is singular, then $Q\underline{x} = \underline{0}$ has a nontrivial solution $\underline{x}$

So $R\underline{x} = (PQ)\underline{x} = P(\underbrace{Q\underline{x}}_{=\underline{0}}) = P\underline{0} = \underline{0}$ has a nontrivial soln, so R is singular

(2) If Q is nonsingular, then P is singular. So we can find $\underline{b} \neq \underline{0}$ with $P\underline{b} = \underline{0}$. Now: $Q\underline{x} = \underline{b}$ has a unique solution ($\underline{x} \neq \underline{0}$ also)

So $R\underline{x} = PQ\underline{x} = P\underline{b} = \underline{0}$. So R is singular.

Consequence: If $AB = I_n$ with A, B of size $n \times n$, then $BA = I_n$.

Why? Use Lemma! $P = A$, $B = Q$, $R = AB = I_n$.

Since R is nonsingular, the Lemma forces A & B to be nonsingular.

In particular, $\underline{B}\ \underset{n\times n}{\underline{X}} = I_n$ admits a unique solution $\underline{X} = \underline{C}$

(we have n systems, with $b = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$)

Then $AB = BC = I_n$. This forces $C = A$ because

$$\boxed{A} = A I_n = A(BC) = (AB)C = I_n C = \boxed{C}.$$

(Trick from last time)

Unit Vectors In $\mathbb{R}^n$ we have n special vectors of norm 1.

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots\ldots, \quad e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

(Standard Unit Vectors)

- We can use them to write any vector! (linear combination)

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = a_1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{= e_1} + a_2 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{= e_2} + \cdots + a_n \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{= e_n}$$

Linear Independence in $\mathbb{R}^n$

Def: • A set of vectors $\{\vec{v}_1, \dots, \vec{v}_p\}$ in $\mathbb{R}^n$ is linearly independent (l.i.) if the only solution $(a_1, \dots, a_p)$ to the vector equation:

$$(*) \quad \boxed{\underbrace{a_1}\vec{v}_1 + \underbrace{a_2}\vec{v}_2 + \dots + \underbrace{a_p}\vec{v}_p = \vec{0} \quad \text{in } \mathbb{R}^n} \qquad \text{unknowns}$$

is the trivial one, that is $a_1 = a_2 = \dots = a_p = 0$.

• If a non-trivial solution to (*) exists, we say the set $\{\vec{v}_1, \dots, \vec{v}_p\}$ is linearly dependent (l.d.) [There is an equation showing they are related!]

Example: ① $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is l.i. (Why? $a_1\vec{v}_1 + a_2\vec{v}_2 = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ forces $a_1 = a_2 = 0$)

② $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is l.d. $\left(1 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + 0 \cdot \vec{v}_3 = \vec{0} \right)$

③ $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$ is l.d. $\left(1 \cdot \vec{v}_1 + 1 \cdot \vec{v}_2 = \vec{0} \right)$

Example: $\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 8 \\ 11 \\ 8 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

① $\boxed{\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \text{ is l.d.}}$

Write

$$\underbrace{a_1 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + a_3 \begin{bmatrix} 8 \\ 11 \\ 8 \end{bmatrix}}_{\begin{array}{c} \begin{bmatrix} 2 & 4 & 8 \\ 3 & 5 & 11 \\ 1 & 6 & 8 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \\ \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \end{array}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Want to solve $A \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\boxed{a_1 = -2, \ a_2 = -1, \ a_3 = 1 \text{ works}}$

$$\begin{bmatrix} 2 & 4 & 8 \\ 3 & 5 & 11 \\ 1 & 6 & 8 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_3]{} \begin{bmatrix} 1 & 6 & 8 \\ 3 & 5 & 11 \\ 2 & 4 & 8 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1}]{} \begin{bmatrix} 1 & 6 & 8 \\ 0 & -13 & -13 \\ 0 & -8 & -8 \end{bmatrix} \xrightarrow[R_2 \to \frac{1}{-13} R_2]{} \begin{bmatrix} 1 & 6 & 8 \\ 0 & 1 & 1 \\ 0 & -8 & -8 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 + 8R_2]{} \begin{bmatrix} 1 & 6 & 8 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1 \to R_1 - 6R_2]{} \underset{\text{REF}}{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}}$$

a_1, a_2 dep
a_3 indep

$\rightsquigarrow \begin{cases} a_1 + 2a_3 = 0 \\ a_2 + a_3 = 0 \\ 0 = 0 \end{cases}$

$\boxed{\begin{array}{l} a_1 = -2a_3 \\ a_2 = -a_3 \\ a_3 \text{ ANY} \end{array}}$

② $\boxed{\{\vec{v}_1, \vec{v}_2, \vec{v}_4\} \text{ is l.i.}}$ $\rightsquigarrow \begin{bmatrix} 2 & 4 & 1 \\ 3 & 5 & 2 \\ 1 & 6 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Q Unique soln?
A YES

$$\begin{bmatrix} 2 & 4 & 1 \\ 3 & 5 & 2 \\ 1 & 6 & 0 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_3]{} \begin{bmatrix} 1 & 6 & 0 \\ 3 & 5 & 2 \\ 2 & 4 & 1 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1}]{} \begin{bmatrix} 1 & 6 & 0 \\ 0 & -13 & 2 \\ 0 & -8 & 1 \end{bmatrix} \xrightarrow[R_2 \to \frac{-1}{13} R_2]{} \begin{bmatrix} 1 & 6 & 0 \\ 0 & 1 & -2/13 \\ 0 & -8 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 8R_1]{} \begin{bmatrix} 1 & 6 & 0 \\ 0 & 1 & -2/13 \\ 0 & 0 & -3/13 \end{bmatrix}$$

$$\xrightarrow[R_3 \to \frac{13}{-3} R_3]{} \begin{bmatrix} 1 & 6 & 0 \\ 0 & 1 & -2/13 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution is unique (rank = 3) $a_1 = a_2 = a_4 = 0$

$\underline{Q}$ Why study independence?

$\underline{\text{Proposition 1}}$: The system $\underset{n\times p}{A} \cdot \underline{x} = \underline{0}$ has a unique solution if and only if the p columns of A are l.i. vectors in $\mathbb{R}^n$.

$\underline{\text{Why}}$? $A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \underline{x_1}\, \mathrm{Col}_1 A + \underline{x_2}\, \mathrm{Col}_2 A + \cdots + \underline{x_p}\, \mathrm{Col}_p A = \underline{0}$ in $\mathbb{R}^n$.

$\underline{\text{Proposition 2}}$: A of size $n\times p$, $\underline{b}$ in $\mathbb{R}^n$. Then, the system $A\,\underline{x} = \underline{b}$ has a solution if and only if $\underline{b}$ is a <u>linear combination</u> of the p columns of A $\left(\underline{x_1}\, \mathrm{Col}_1(A) + \underline{x_2}\, \mathrm{Col}_2(A) + \cdots + \underline{x_p}\, \mathrm{Col}_p(A) = \underline{b}\right)$

The solution is <u>unique</u> only when the columns of A are l.i.

$\underline{\text{Theorem}}$: If $\{\vec{v}_1, \ldots, \vec{v}_p\}$ are l.d. with $a_1\vec{v}_1 + \cdots + a_p\vec{v}_p = \underline{0}$ & $a_i \neq 0$, then $\vec{v}_i$ is a linear comb. of the remaining vectors.

$\underline{\text{Why}}$? If $i=1$, $a_1\vec{v}_1 = (-a_2)\vec{v}_2 + \cdots + (-a_p)\vec{v}_p$

$$\boxed{\vec{v}_1 = (-a_2/a_1)\,\vec{v}_2 + \cdots + (-a_p/a_1)\,\vec{v}_p}$$

Useful Properties

$e_1\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, e_2\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}, \dots, e_n = \begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}$

<u>Prop 0</u>: Standard unit Vectors are li $\left([e_1 \cdots e_n] = I_n \right)$

<u>Prop 1</u>: Reordering vectors preserves linear independency / dep.

<u>Prop 2</u>: A subset of a li set is li

<u>Why</u>? $\{\vec{v}_1, \dots \vec{v}_p\}$ is li but $\{v_1, \dots v_s\}$ ld, then

$a_1\vec{v}_1 + \dots + a_s\vec{v}_s + 0\cdot\vec{v}_{s+1} + \dots + 0\,\vec{v}_p = 0$ gives ld. <u>Contr</u>!

<u>Ex</u>: $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$ is li, so $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$, $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$ & $\left\{ \begin{bmatrix}0\\1\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$ are also li.

<u>Theorem 3</u>: If $\{\vec{v}_1, \dots \vec{v}_p\}$ in $\mathbb{R}^n$ & $p > n$, then $\{\vec{v}_1, \dots \vec{v}_p\}$ is ld.

<u>Why</u>? We want to solve $\underbrace{\begin{bmatrix} v_1 & \cdots & v_p \end{bmatrix}}_{n \times p} \begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ in $\mathbb{R}^n$

$n = \#$ Eqns $< \#$ Vars $= p$ so we can't have a unique soln.

<u>Theorem 4</u>: A matrix is non-singular if and only if cols are li

Summary of Results

Fix A of size $n \times n$. The following statements are equivalent:

1. A is nonsingular ($A\underline{x} = \underline{0}$ has a unique solution)
2. The n columns of A are li
3. $A\underline{x} = \underline{b}$ has a unique solution for all $\underline{b}$ in $\mathbb{R}^n$.
4. A is invertible
5. A is row equivalent to I_n ($[A|I_n] \sim [I_n|B]$) in ALGORITHM for finding A^{-1}

Example $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

1. $\begin{cases} x_1 + x_2 = 0 \\ x_2 = 0 \end{cases}$ forces $x_1 = x_2 = 0$
2. $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ are li
3. $\begin{cases} x_1 + x_2 = b_1 \\ x_2 = b_2 \end{cases} \sim \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 - b_2 \\ b_2 \end{bmatrix}$
4. $A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$
5. $A \sim I_2$