# Lecture 10: §2.1 Vectors in the Plane & §2.2 Vectors in Space

**Recall**: Vectors in $\mathbb{R}^n$ $=$ $n \times 1$ matrices

In $\mathbb{R}^2$: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, etc. In $\mathbb{R}^3$: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, etc.

**TODAY**: How can we represent a vector geometrically?

Why? Use vectors to model Physics (e.g. force, displacement, velocity, etc...)

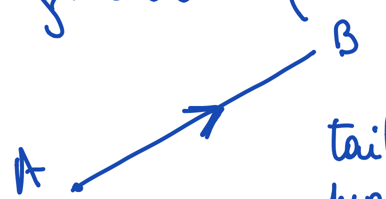
**Q.** What is a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$?

**A.** Informally, a vector represents 2 pieces of data

1. magnitude (or length)
2. direction

We depict them by a directed line segment (= an arrow!)

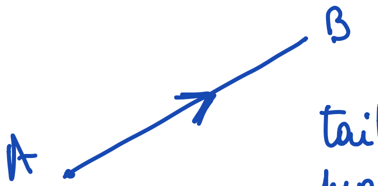
$\begin{cases} \text{length of the segment} = \text{magnitude} = |\vec{v}| \\ \text{arrow : indicates direction} \end{cases}$




$\vec{v} = \overrightarrow{AB}$

tail = A = initial pt

head = B = terminal pt.



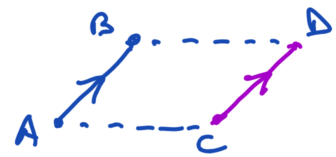

$\vec{v} = \vec{AB}$

tail = A = initial pt

head = B = terminal pt.

## Observations:

① We do not distinguish between parallel vectors of the same length

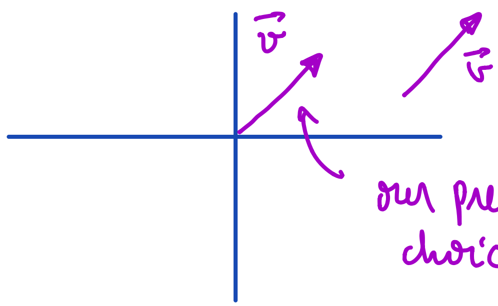


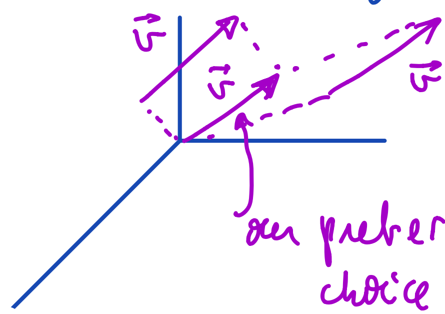

$\vec{AB}$ & $\vec{CD}$ represent the same vector.

② $\vec{0}$ = • is the only vector of magnitude 0. It has **no** direction.

**Q**: Can we have a consistent choice to represent all these vectors?

**A**: YES! Always draw the tail at the origin = O ( O = (0,0) in $\mathbb{R}^2$, O = (0,0,0) in $\mathbb{R}^3$ )

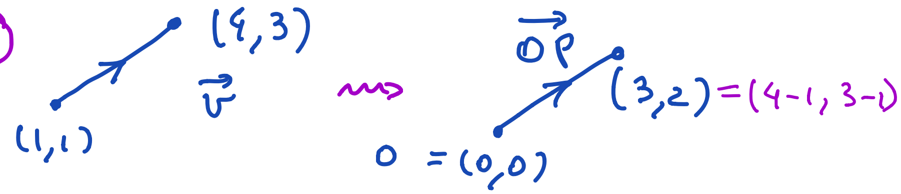


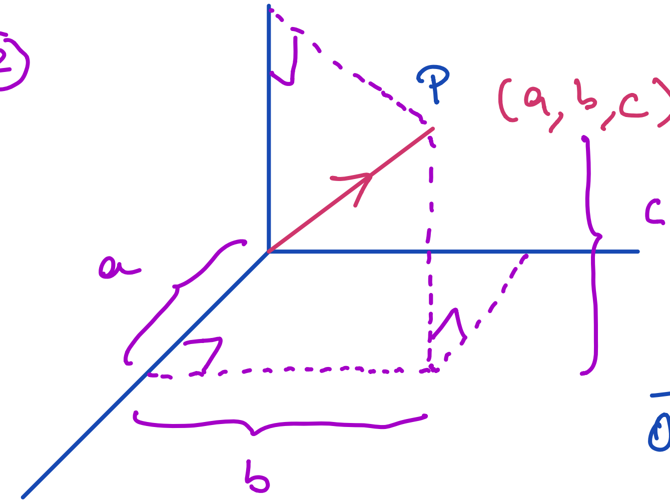

Name: position vector

After this choice is made, we can identify the vector with the location of its head = P. Write $\vec{v} = \vec{OP}$ (position vector). Coordinates of P are called the components of $\vec{v}$.

# Examples

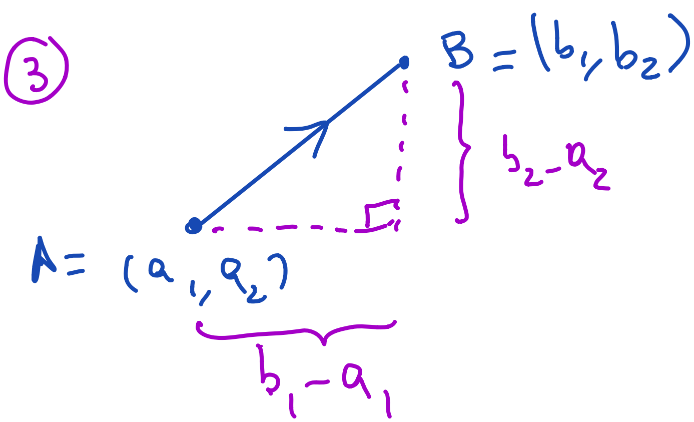
① 


position vector

② 


$\overrightarrow{OP} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

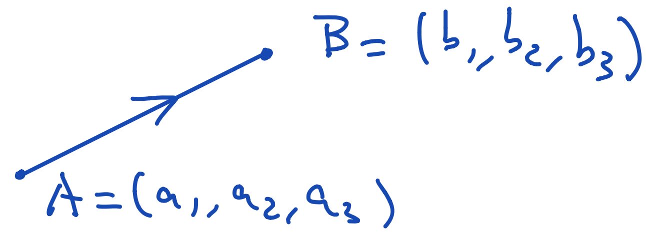
3 right angles

③ 


$\rightsquigarrow$ x-component of $\vec{v}$ $= b_1 - a_1$

y-component of $\vec{v}$ $= b_2 - a_2$

$\overrightarrow{OP} = \begin{bmatrix} b_1 - a_1 \\ b_2 - a_2 \end{bmatrix}$ position vector for $\overrightarrow{AB}$

④ 


$\rightsquigarrow$ 3 components:

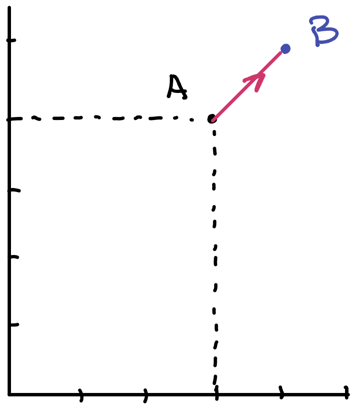
$$\begin{cases} x\text{-comp} = b_1 - a_1 \\ y\text{-comp} = b_2 - a_2 \\ z\text{-comp} = b_3 - a_3 \end{cases} \rightsquigarrow \overrightarrow{OP} = \begin{bmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{bmatrix}$$

position vector

**Q**: Why use position vectors?

**A** We can decide when $\vec{v} = \vec{w}$! They must have the SAME position vector

**Example**: Given $v = \overrightarrow{AB}$ with position vector $\begin{bmatrix}1\\1\end{bmatrix}$, find the coordinates of B given $A = (3,4)$ & draw $\vec{v}$




$B = (b_1, b_2)$
$A = (3,4)$

$$\rightsquigarrow \begin{bmatrix}1\\1\end{bmatrix} = \overrightarrow{OP} = \begin{bmatrix}b_1-3\\b_2-4\end{bmatrix}, \text{ so } \begin{array}{l} b_1=4 \\ b_2=5 \end{array}$$

$$\boxed{B=(4,5)}$$

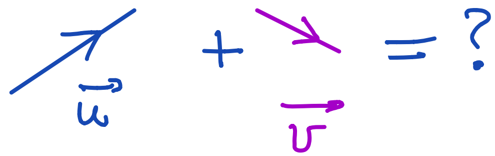
Recall: Magnitude of $\vec{v}$ $= \begin{cases} \sqrt{v_1^2+v_2^2} & \text{if } \vec{v} = \begin{bmatrix}v_1\\v_2\end{bmatrix} \text{ in } \mathbb{R}^2 \\ \sqrt{v_1^2+v_2^2+v_3^2} & \text{if } \vec{v} = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} \text{ in } \mathbb{R}^3 \end{cases}$

**Operations on vectors**: ① Addition ② Scalar Multiplication } Easy to do algebraically (they are $n \times 1$ matrices!)

**Q**: How to do them geometrically?

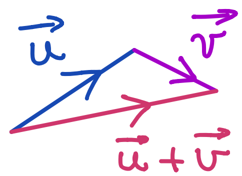
- See these operations for $\mathbb{R}^2$, but the same idea works for $\mathbb{R}^3$.

## (1) ADDITION

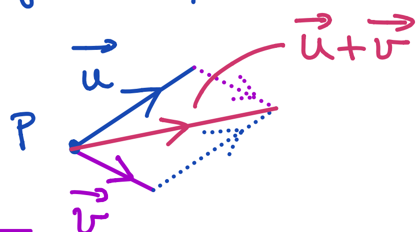



Two ways:

- <u>Triangle Law</u>: Put Tail of $\vec{v}$ at the head of $\vec{u}$ & draw $\Delta$.



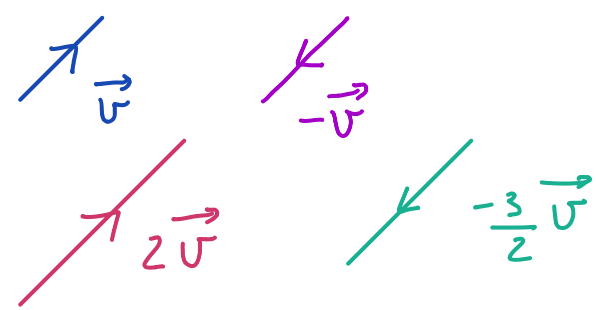

- <u>Parallelogram Law</u>: Put 2 tails at the same point P, draw parallelogram & the diagonal from P to opposite corner.




## (2) SCALAR MULT.

$\left.\begin{array}{ll}\vec{v} & \text{vector} \\ \alpha & \text{scalar}\end{array}\right\} \rightsquigarrow \alpha\vec{v}$ is a vector with

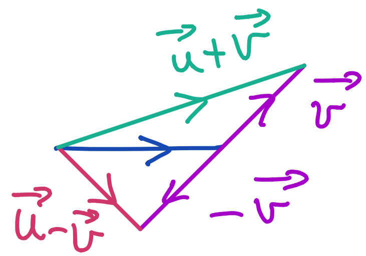
- magnitude $(\alpha\vec{v}) = |\alpha| \, \|\vec{v}\|$
- direction $(\alpha\vec{v}) = \begin{cases} \text{none} & \text{if } \alpha = 0 \\ \text{same as } \vec{v} & \text{if } \alpha > 0 \\ \text{opposite to } \vec{v} & \text{if } \alpha < 0 \end{cases}$



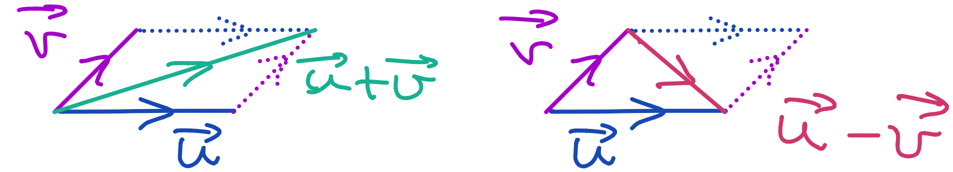
③ Substraction / Difference = Combine ① & ②.

$$\vec{u} - \vec{v} = \vec{u} + (-1)\cdot\vec{v}$$


Triangle Law:



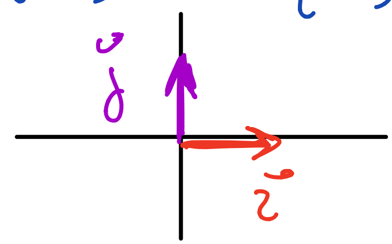

Parallelogram Law : Take the $2^{nd}$ diagonal!



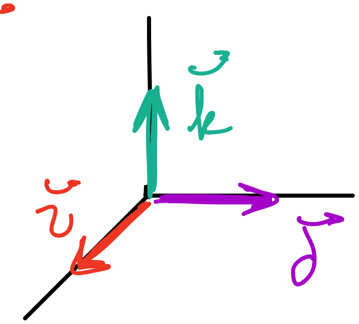

Recall: Basic unit vectors (in $\mathbb{R}^n$)

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots\dots, e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

For $\mathbb{R}^2$ : $\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$




For $\mathbb{R}^3$ : $\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$




**Proposition** Any $\vec{v}$ in $\mathbb{R}^n$ is a linear combination of basic unit vectors

Ex : $\begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ; $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

__Example__ Solve $a\begin{bmatrix}1\\2\\3\end{bmatrix}+b\begin{bmatrix}4\\0\\-1\end{bmatrix}=\begin{bmatrix}1\\-2/3\\-4/3\end{bmatrix}$ for $a, b$ in $\mathbb{R}$

(LHS) reads
$$\begin{aligned} a+4b &= a+4b\\ 2a+0\cdot b &= 2a\\ 3a-b &= 3a-b\end{aligned}$$

Need to solve the system $\begin{cases} a+4b=1\\ 2a=-2/3\\ 3a-b=-4/3\end{cases}$ $\rightsquigarrow$ $A=\begin{bmatrix}1&4\\2&0\\3&-1\end{bmatrix}$

$$B=\left[\begin{array}{cc|c}1&4&1\\2&0&-2/3\\3&-1&-4/3\end{array}\right]\xrightarrow[R_1\Leftrightarrow R_2]{R_2\to\frac{1}{2}R_2}\left[\begin{array}{cc|c}1&0&-1/3\\1&4&1\\3&-1&-4/3\end{array}\right]\xrightarrow[R_3\to R_3-3R_1]{R_2\to R_2-R_1}\left[\begin{array}{cc|c}1&0&-1/3\\0&4&4/3\\0&-1&-1/3\end{array}\right]\xrightarrow[R_2\to -R_2]{R_2\Leftrightarrow R_3}\left[\begin{array}{cc|c}1&0&-1/3\\0&1&1/3\\0&4&4/3\end{array}\right]$$

$$\xrightarrow[R_3\to R_3-4R_2]{}\left[\begin{array}{cc|c}1&0&-1/3\\0&1&1/3\\0&0&0\end{array}\right]$$ REF

Unique soln: $\boxed{\begin{aligned}a&=-\frac{1}{3}\\ b&=\frac{1}{3}\end{aligned}}$

__Check__: $-\frac{1}{3}\begin{bmatrix}1\\2\\3\end{bmatrix}+\frac{1}{3}\begin{bmatrix}4\\0\\-1\end{bmatrix}=\begin{bmatrix}1\\-2/3\\-4/3\end{bmatrix}$ ✓

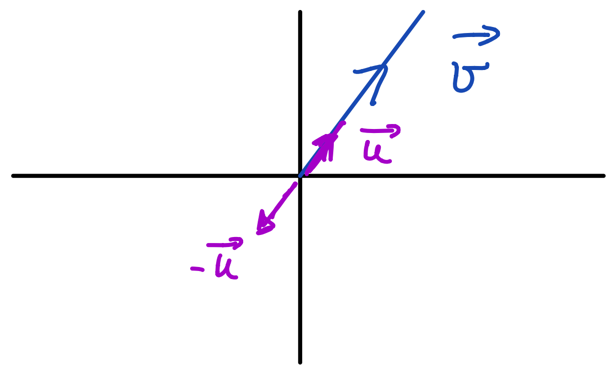
__System__: $\begin{bmatrix}1&4\\2&0\\3&-1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}1\\-2/3\\-4/3\end{bmatrix}$

**Example 1** Find all vectors of length 1 parallel to $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

**Soln**: Vectors are parallel so $\vec{u} = a\vec{v} = \begin{bmatrix} 3a \\ 4a \end{bmatrix}$ for some $a$

$$1 \overset{?}{=} \|\vec{u}\| = \sqrt{(3a)^2 + (4a)^2} = \sqrt{(9+16)a^2} = |a|\sqrt{25} = |a|5$$

Forces $|a| = \frac{1}{5}$ so $a = \pm\frac{1}{5}$

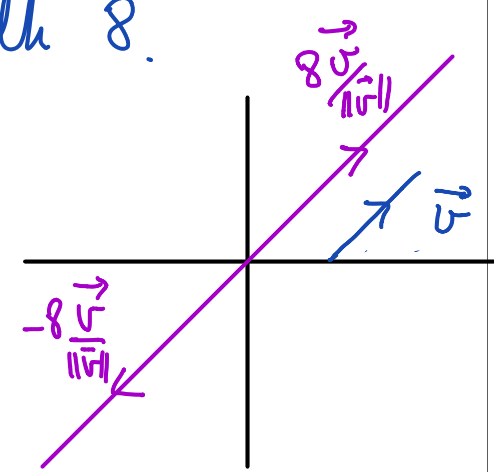



**Answer**: $\begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$ & $-\begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$

**Note**: $\|\vec{v}\| = 5$ so $a = \pm\frac{1}{5} = \pm\frac{1}{\|\vec{v}\|}$

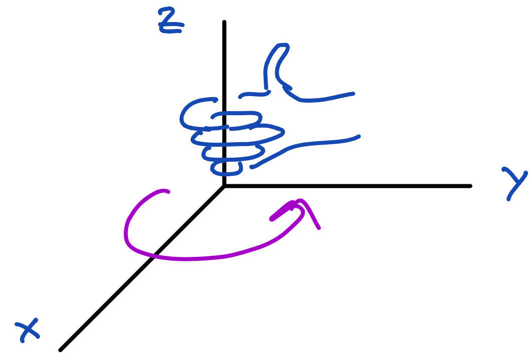
**Example 2**: Same question, but now require length 8.

**A**: $\vec{u} = \pm 8\frac{\vec{v}}{\|\vec{v}\|} = \pm 8\begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} = \pm\begin{bmatrix} 24/5 \\ 32/5 \end{bmatrix}$



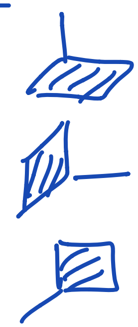
# Rectangular Coordinates in $\mathbb{R}^3$

The three coordinate axes are directed according to the right hand rule

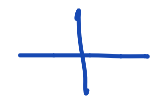



3 coordinate planes :

| Plane | Which axes | Equation |
|---|---|---|
| XY | x- & y- | $z=0$ |
| XZ | x & z- | $Y=0$ |
| YZ | y & z- | $X=0$ |



. These planes subdivide $\mathbb{R}^3$ into 8 octants (for $\mathbb{R}^2$ we had 4)

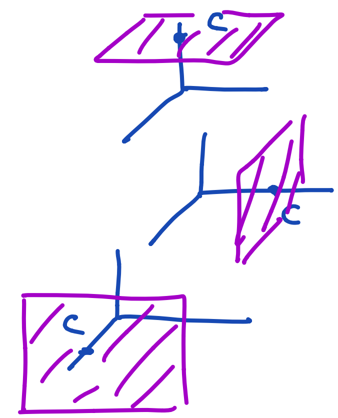


Planes parallel to coordinate planes =

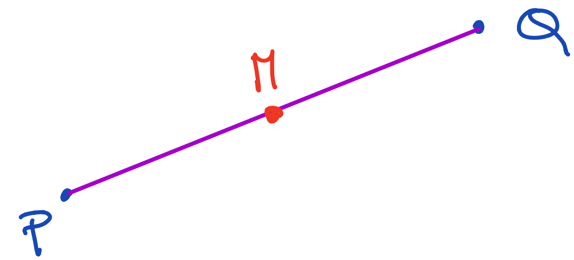
(1) $z=c$

(2) $Y=c$

(3) $X=c$



Q: Midpoint between $P$ & $Q$?




$P = (p_1, p_2, p_3)$

$Q = (q_1, q_2, q_3)$

$M$ has 2 properties:

① $M$ in the segment joining $P$ & $Q$

$\vec{PM}$, $\vec{MQ}$ both parallel to $\vec{PQ}$

$$\begin{cases} \vec{PM} = a\,\vec{PQ} \\ \vec{MQ} = b\,\vec{MQ} \end{cases}$$

② distance $(P, M)$ = distance $(M, Q)$ $= \frac{1}{2}$ distance $(P, Q)$

So $\|\vec{PM}\| = \frac{1}{2}\|\vec{PQ}\| = |a|\,\|\vec{PQ}\|$ forces $a = \pm\frac{1}{2}$

$\|\vec{MQ}\| = \frac{1}{2}\|\vec{PQ}\| = |b|\,\|\vec{PQ}\|$ forces $b = \pm\frac{1}{2}$

But $\vec{PM}$ & $\vec{MQ}$ have the same direction as $\vec{PQ}$ so $a = b = \frac{1}{2}$

<u>Conclude</u> $M = (m_1, m_2, m_3)$ satisfies

$$m_1 - p_1 = \tfrac{1}{2}(q_1 - p_1)$$
$$m_2 - p_2 = \tfrac{1}{2}(q_2 - p_2)$$
$$m_3 - p_3 = \tfrac{1}{2}(q_3 - q_3)$$

So $\boxed{M = \left(\frac{q_1 + p_1}{2}, \frac{q_2 + p_2}{2}, \frac{q_3 + p_3}{2}\right)}$