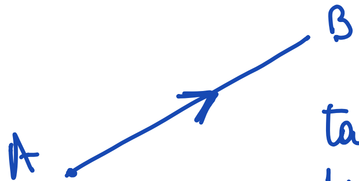
Lecture 11: §2.3 The Dot & Cross Products

Recall: Vectors in $\mathbb{R}^n$ $=$ $n \times 1$ matrices

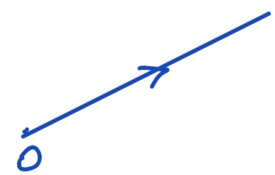
In $\mathbb{R}^2$ or $\mathbb{R}^3$: directed arrow (length $=$ magnitude of the vector $= \|\vec{v}\|$)



$\vec{v} = \overrightarrow{AB}$

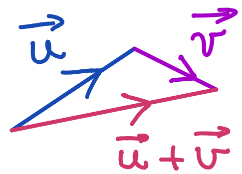
tail $= A =$ initial pt

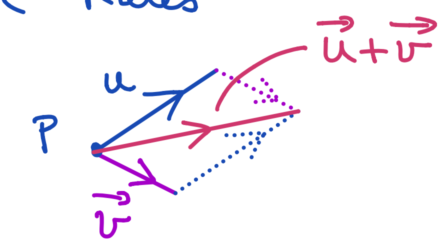
head $= B =$ terminal pt.

Position vector: 

$P =$ "$B - A$"

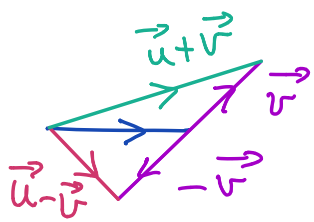
Saw how to add & scalar multiply vectors with geometric Rules

① • Triangle Law: 

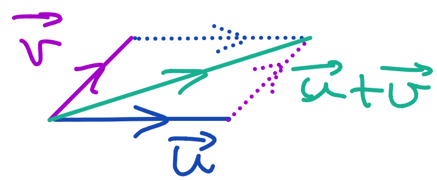
• Parallelogram Law: 

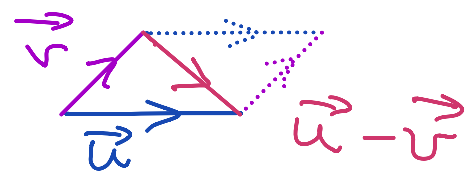
② $\alpha \vec{v}$ has $\|\alpha \vec{v}\| = |\alpha| \, \|\vec{v}\|$ & direction $(\alpha \vec{v}) = \begin{cases} \text{none} & \text{if } \alpha = 0 \\ \text{same as } \vec{v} & \text{if } \alpha > 0 \\ \text{opposite to } \vec{v} & \text{if } \alpha < 0 \end{cases}$

Substraction / Difference $=$ combine ① & ② $\left(\vec{u} - \vec{v} = \vec{u} + (-1) \cdot \vec{v}\right)$

Triangle Law: 

Parallelogram Law: Take the 2^{nd} diagonal!





The Dot Product

Recall Given $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$, we define the dot product

$\vec{u} \cdot \vec{v} =$

Algebraic Properties: Inherited from matrix operations & their properties

Fix $\vec{u}, \vec{v}, \vec{w}$ vectors, α scalar.

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ (not true for general matrices!)
2. $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w} = \vec{w} \cdot (\vec{u} + \vec{v})$ [Distributive]
3. $(\alpha \vec{u}) \cdot \vec{v} = \alpha (\vec{u} \cdot \vec{v}) = \vec{u} \cdot (\alpha \vec{v})$ [Associative]
4. $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$

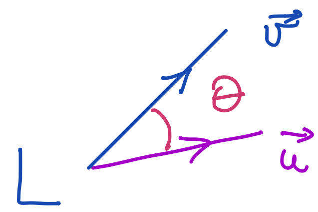
Q: Can we compute $\vec{u} \cdot \vec{v}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ using Geometry?

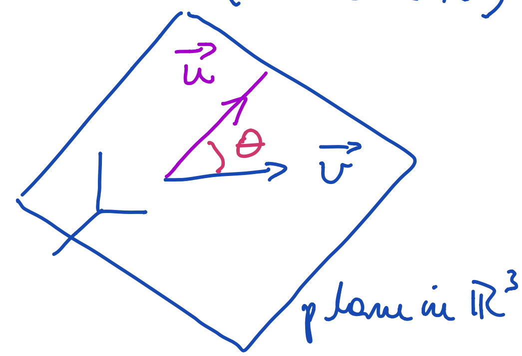
A: YES!

$$\boxed{\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_m v_m}$$

Geometric fn of $\vec{u} \cdot \vec{v}$: Fix $\vec{u}, \vec{v}$ & θ = angle between them $(0 \le \theta \le \pi)$

Theorem: $\boxed{\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta}$





$$\boxed{\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta}$$

Example 1: Pick $\vec{u}, \vec{v}$ of length 3 & 6, respectively with angle $\theta = 60^\circ$ between them. Find $\vec{u}\cdot\vec{v}$.

Example 2: Find the angle between $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ & $\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$

Observation: $\vec{0}\cdot\vec{u} = 0$ for all $\vec{u}$.

Definition: Two vectors are perpendicular (or orthogonal) whenever the angle between them is 90°. Write $\vec{u} \perp \vec{v}$.

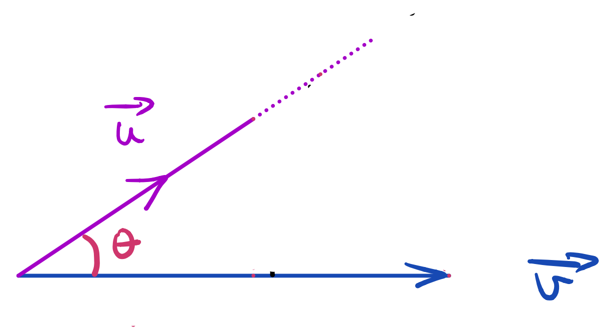
Orthogonal Projections

(Key Step in Gramm-Schmidt §3.6)

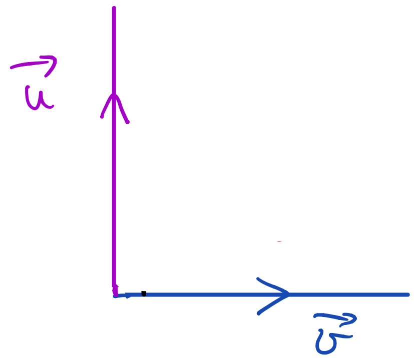
Fix $\vec{u}, \vec{v} \neq \vec{0}$

① $\text{proj}_{\vec{v}}\vec{u}$ = orthogonal projection of $\vec{u}$ onto $\vec{v}$ (vector projection of $\vec{u}$ along $\vec{v}$)

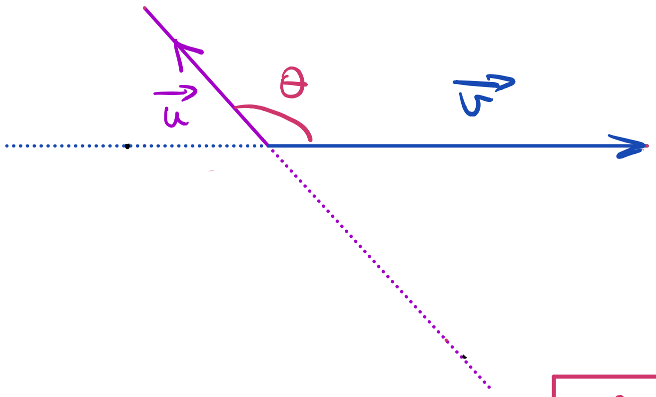
② $\text{proj}_{\vec{u}}\vec{v}$ = orthogonal projection of $\vec{v}$ onto $\vec{u}$ (vector projection of $\vec{v}$ along $\vec{u}$)



$0 \leq \theta < 90°$



$\theta = 90°$



$90° < \theta \leq 180°$

$$\boxed{\operatorname{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\,\vec{v}}$$

$$\boxed{\text{Signed Length} = \operatorname{comp}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}}$$

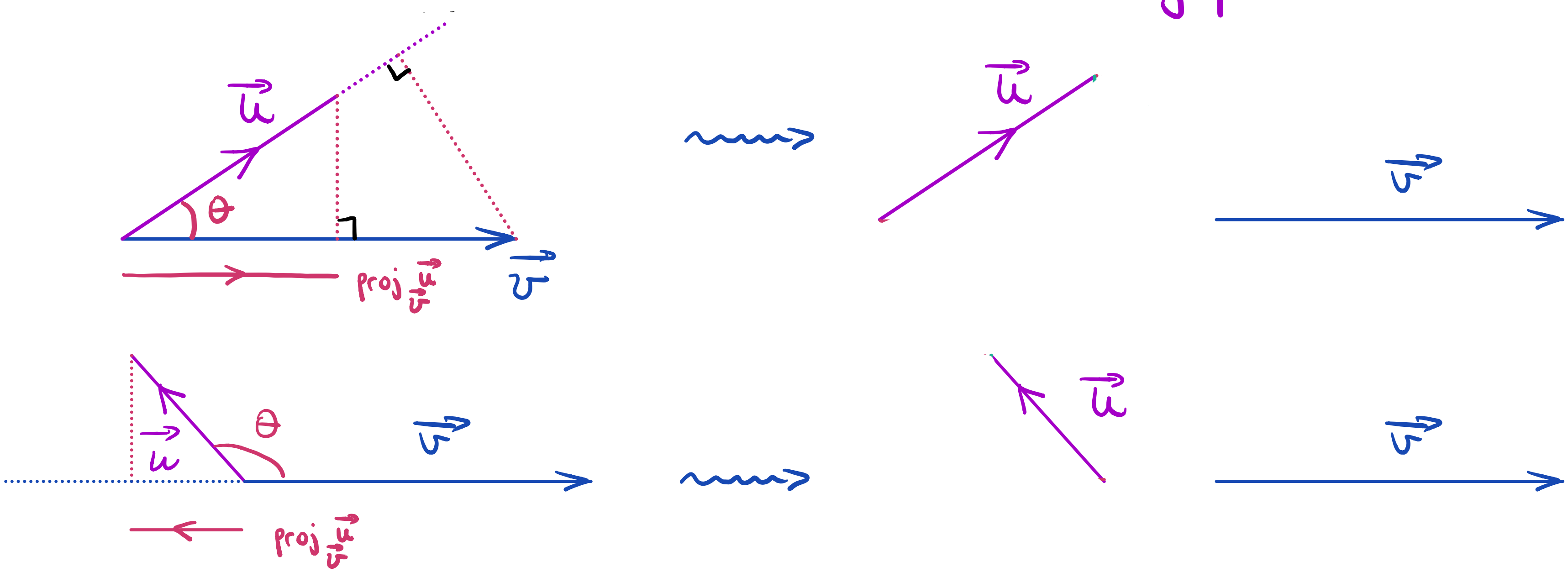
Q: Why is this important?

A We can decompose $\vec{u}$ as a sum $\vec{w} + \vec{s}$

$$\boxed{\vec{u} = \underbrace{\operatorname{proj}_{\vec{v}}\vec{u}}_{\vec{w}} + \underbrace{\left(u - \operatorname{proj}_{\vec{v}}\vec{u}\right)}_{\vec{s}}}$$

where $\vec{w} \parallel \vec{v}$, $\vec{s} \perp \vec{v}$

$\vec{s}$ ⇝ key for Gramm-Schmidt

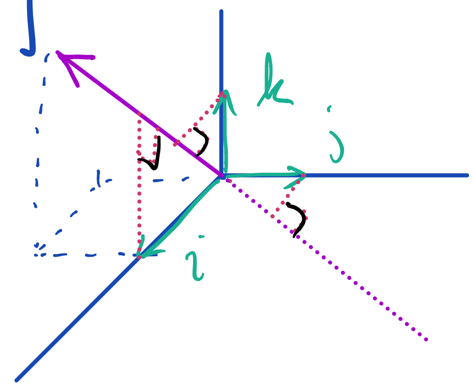


Key: $\vec{w}$ & $\vec{s}$ are the only vectors with these properties.

$$\boxed{\operatorname{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v}}$$

$$\boxed{\operatorname{comp}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|v\|}}$$

Example: Compute the projections of $\vec{i}, \vec{j}, \vec{k}$ along $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$



$\boxed{\text{Cross Product (in } \mathbb{R}^3)}$

Fix $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$. The <u>cross product</u> $\vec{u} \times \vec{v}$ is a vector in $\mathbb{R}^3$ with coordinates:

$$\vec{u} \times \vec{v} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$$\boxed{\vec{u} \times \vec{v} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} = \underbrace{\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}}_{=\text{scalar}} \vec{i} - \underbrace{\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}}_{=\text{scalar}} \vec{j} + \underbrace{\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}}_{=\text{scalar}} \vec{k}}$$

Example: $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\quad \vec{v} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ $\quad \vec{u} \times \vec{v} = \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix}$ since

Properties: $\vec{u}, \vec{v}, \vec{w}$ in $\mathbb{R}^3$, α, β scalars

1. $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$ [ANTICOMMUTATIVE], so $\vec{u} \times \vec{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ for all $\vec{u}$
2. $\alpha\vec{u} \times \beta\vec{v} = (\alpha\beta)(\vec{u} \times \vec{v})$ [Associative], so $\vec{0} \times \vec{u} = \vec{0}$
3. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
4. $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$

   [Distributive] (for 3 and 4)

5. $\vec{u} \cdot (\vec{v} \times \vec{w}) = (u \times \vec{v}) \cdot \vec{w}$

Q: Why anticommutative? $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$.

Key Proposition: $\vec{u} \times \vec{v} \perp \vec{u}$ & $\vec{u} \times \vec{v} \perp \vec{v}$