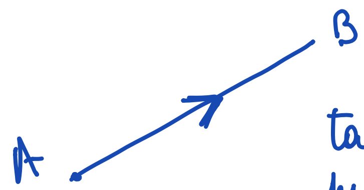
# Lecture 11: §2.3 The Dot & Cross Products

**Recall:** Vectors in $\mathbb{R}^n$ $=$ $n \times 1$ matrices

In $\mathbb{R}^2$ or $\mathbb{R}^3$: directed arrow (length $=$ magnitude of the vector $= \|\vec{v}\|$)

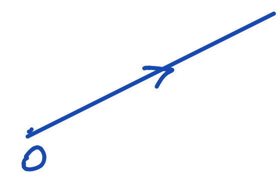



$\vec{v} = \overrightarrow{AB}$

tail $=$ A $=$ initial pt

head $=$ B $=$ terminal pt.

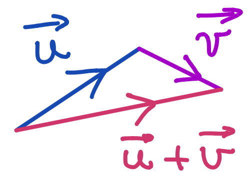
Position vector:



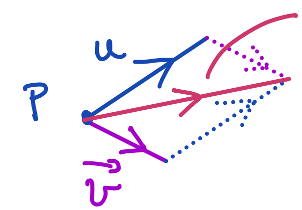

P $=$ "B−A"

Saw how to add & scalar multiply vectors with geometric rules

① • <u>Triangle Law</u>:



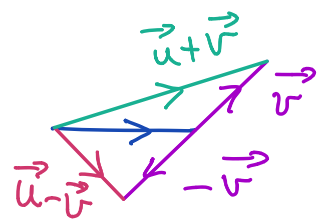

• <u>Parallelogram Law</u>:



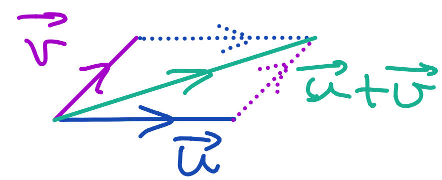

② $\alpha\vec{v}$ has $\|\alpha\vec{v}\| = |\alpha|\,\|\vec{v}\|$ & direction$(\alpha\vec{v}) = \begin{cases} \text{none} & \text{if } \alpha = 0 \\ \text{same as } \vec{v} & \text{if } \alpha > 0 \\ \text{opposite to } \vec{v} & \text{if } \alpha < 0 \end{cases}$

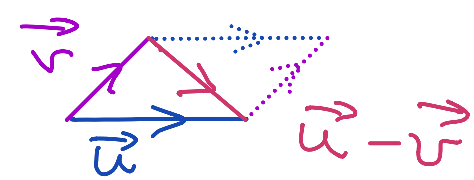
Substraction / Difference $=$ combine ① & ② $(\vec{u} - \vec{v} = \vec{u} + (-1)\cdot\vec{v})$

<u>Triangle Law</u>:




<u>Parallelogram Law</u>: Take the $2^{nd}$ diagonal!



# The Dot Product

Recall Given $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$ in $\mathbb{R}^m$, we define the dot product

$$\vec{u} \cdot \vec{v} = \underset{1\times n}{\vec{u}^T} \; \underset{n\times 1}{\vec{v}} = u_1 v_1 + u_2 v_2 + \cdots + u_m v_m \quad \text{(a number!)}$$

Example: $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}$ $\rightsquigarrow$ $\vec{u} \cdot \vec{v} = 1 \cdot 0 + (-1) \cdot 3 + 1 \cdot 5 = -3 + 5 = \boxed{2}$

Algebraic Properties: Inherited from matrix operations & their properties

Fix $\vec{u}, \vec{v}, \vec{w}$ vectors, $\alpha$ scalar.

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ (not true for general matrices!)
2. $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w} = \vec{w} \cdot (\vec{u} + \vec{v})$ [Distributive]
3. $(\alpha \vec{u}) \cdot \vec{v} = \alpha (\vec{u} \cdot \vec{v}) = \vec{u} \cdot (\alpha \vec{v})$ [Associative]
4. $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$

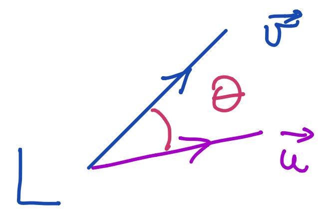
Q: Can we compute $\vec{u} \cdot \vec{v}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ using Geometry?

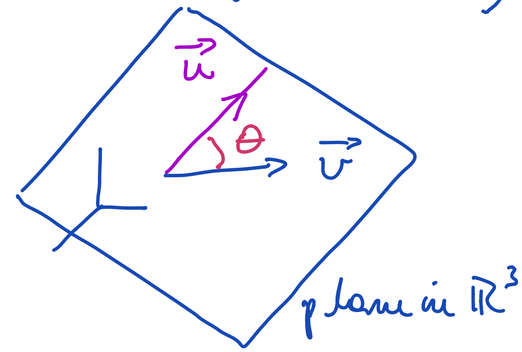
A: YES!

$$\boxed{\vec{u}\cdot\vec{v} = \vec{u}^T\,\vec{v} = u_1v_1 + u_2v_2 + \cdots + u_mv_m}$$

**Geometric for of $\vec{u}\cdot\vec{v}$:** Fix $\vec{u}, \vec{v}$ & $\theta$ = angle between them $(0 \le \theta \le \pi)$

**Theorem:** $\boxed{\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta}$




**Why?** Write $a := \|\vec{u}\|$ $b := \|\vec{v}\|$

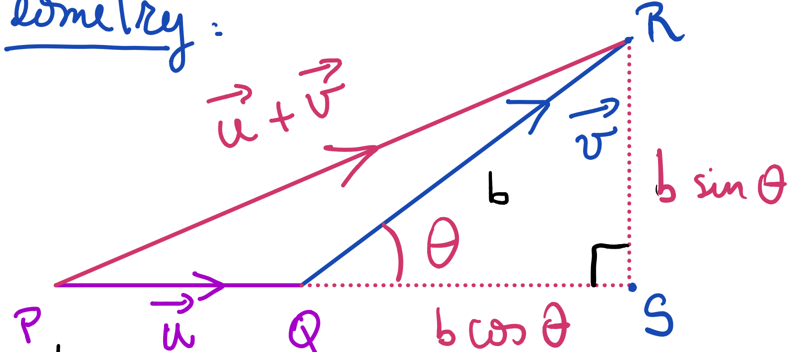
We compute $\|\vec{u}+\vec{v}\|^2$ in 2 ways:

- **Algebra:** $\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = \vec{u}\cdot\vec{u} + \underbrace{\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u}}_{2\,\vec{u}\cdot\vec{v}} + \vec{v}\cdot\vec{v}$

$$= \|\vec{u}\|^2 + 2\,\vec{u}\cdot\vec{v} + \|\vec{v}\|^2$$

$$= a^2 + b^2 + \boxed{2\,\vec{u}\cdot\vec{v}} \quad (\mathrm{I})$$

- **Geometry:**




$\rightsquigarrow$ Boxes agree!

$$\|\vec{u}+\vec{v}\|^2 = |PR|^2$$

$$= |PS|^2 + |SR|^2$$

$$= (a + b\cos\theta)^2 + (b\sin\theta)^2$$

$$= a^2 + 2ab\cos\theta + \underbrace{b^2\cos^2\theta + b^2\sin^2\theta}_{= b^2}$$

$$= a^2 + b^2 + \boxed{2ab\cos\theta} \quad (\mathrm{II})$$

$$\boxed{\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta}$$

**Example 1:** Pick $\vec{u}, \vec{v}$ of length 3 & 6, respectively with angle $\theta = 60°$ between them. Find $\vec{u}\cdot\vec{v}$.

**Soln:** $\vec{u}\cdot\vec{v} = 3\cdot 6\cdot\cos 60° = \frac{18}{2} = \boxed{9}$

**Example 2:** Find the angle between $\vec{u} = \begin{bmatrix}2\\1\\0\end{bmatrix}$ & $\vec{v} = \begin{bmatrix}-1\\1\\5\end{bmatrix}$

**Soln:** $\|\vec{u}\| = \sqrt{2^2+1^2+0^2} = \sqrt{5}$

$\|\vec{v}\| = \sqrt{(-1)^2+1^2+5^2} = \sqrt{27}$

$\vec{u}\cdot\vec{v} = 2(-1)+1\cdot 1+0\cdot 5$
$= \boxed{-1}$

$\rightsquigarrow \cos\theta = \frac{-1}{\sqrt{5}\sqrt{27}} = \frac{-1}{3\sqrt{15}} \rightsquigarrow \theta \approx 95°.$

**Observation:** $\vec{0}\cdot\vec{u} = 0$ for all $\vec{u}$.

**Definition:** Two vectors are perpendicular (or orthogonal) whenever the angle between them is $90°$. Write $\vec{u} \perp \vec{v}$.

**Consequence:** Assume $\vec{u}, \vec{v} \neq \vec{0}$. Then, $\vec{u} \perp \vec{v}$ if and only if $\vec{u}\cdot\vec{v} = 0$

( Why? $\cos\theta = 0$ & $0 \le \theta \le \pi$ has only one solution: $\theta = \frac{\pi}{2}$ $(=90°)$)

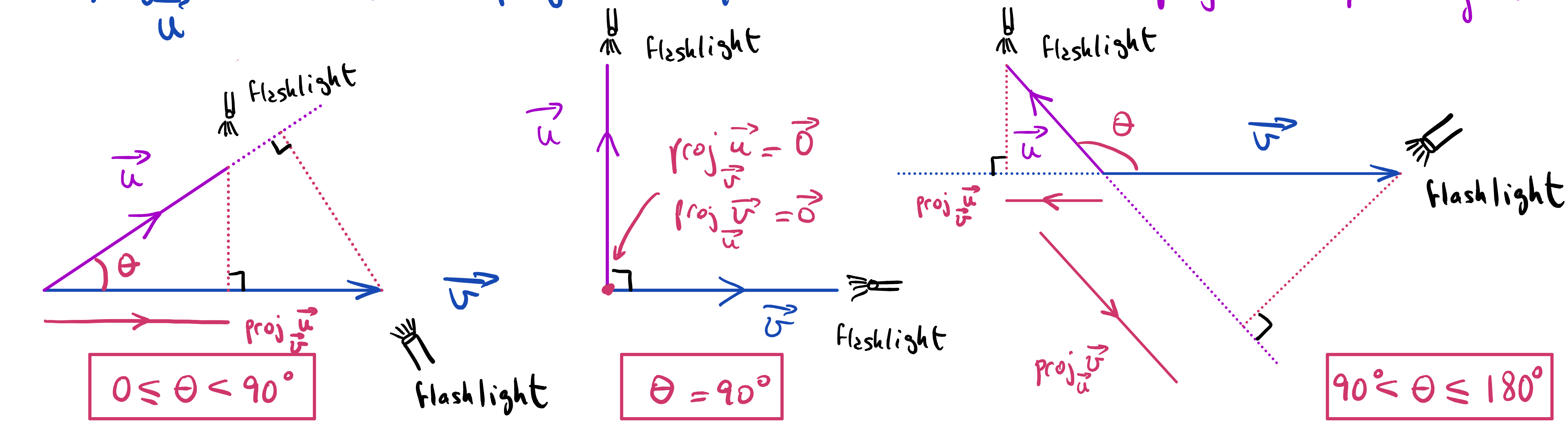
# Orthogonal Projections

(Key Step in Gramm-Schmidt §3.6)

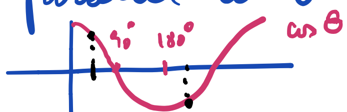
Fix $\vec{u}, \vec{v} \neq \vec{0}$

① $\text{proj}_{\vec{v}}\vec{u}$ = orthogonal projection of $\vec{u}$ onto $\vec{v}$ (vector projection of $\vec{u}$ along $\vec{v}$)

② $\text{proj}_{\vec{u}}\vec{v}$ = orthogonal projection of $\vec{v}$ onto $\vec{u}$ (vector projection of $\vec{v}$ along $\vec{u}$)




① $\text{proj}_{\vec{v}}\vec{u}$ is parallel to $\vec{v}$, so direction is
$$\begin{cases} \text{none} & \text{if } \theta = 90° \quad (\text{vector is } \vec{0}) \\ \pm\dfrac{\vec{v}}{\|\vec{v}\|} & (+ \text{ if } 0 \leq \theta < 90°;\ - \text{ if } 90° < \theta \leq 180°) \end{cases}$$




② $$\|\text{proj}_{\vec{v}}\vec{u}\| = \|\vec{u}\|\,|\cos\theta| = \|\vec{u}\|\,\frac{|\vec{u}\cdot\vec{v}|}{\|\vec{u}\|\,\|\vec{v}\|} = \frac{|\vec{u}\cdot\vec{v}|}{\|\vec{v}\|} = \begin{cases} \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|} & \text{if } 0 < \theta \leq 90° \\ -\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|} & \text{if } 90° < \theta \leq 180° \end{cases}$$

Conclude: $$\boxed{\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\,\vec{v}}$$ $$\boxed{\text{Signed Length} = \text{comp}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}}$$

$$\boxed{\text{proj}_{\vec v}\vec u = \frac{\vec u\cdot\vec v}{\|\vec v\|^2}\,\vec v}$$

$$\boxed{\text{Signed Length} = \text{comp}_{\vec v}\vec u = \frac{\vec u\cdot\vec v}{\|\vec v\|}}$$

**Q:** Why is this important?

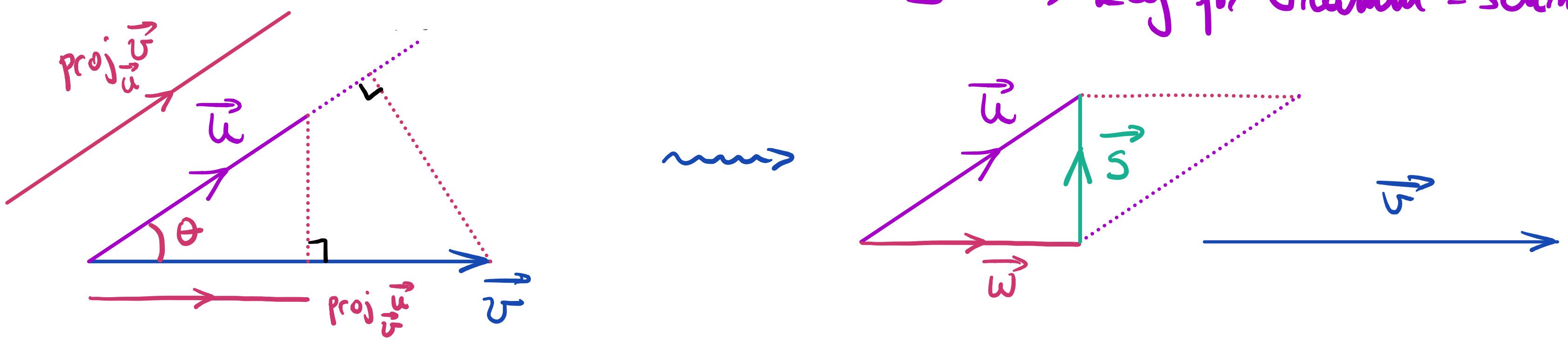
**A:** We can decompose $\vec u$ as a sum $\vec w + \vec s$

$$\boxed{\vec u = \underbrace{\text{proj}_{\vec v}\vec u}_{\vec w} + \underbrace{\left(u - \text{proj}_{\vec v}\vec u\right)}_{\vec s}}$$

where $\vec w \,//\, \vec v$

$\vec s \perp \vec v$

$\vec s$ ⇝ key for Gramm-Schmidt

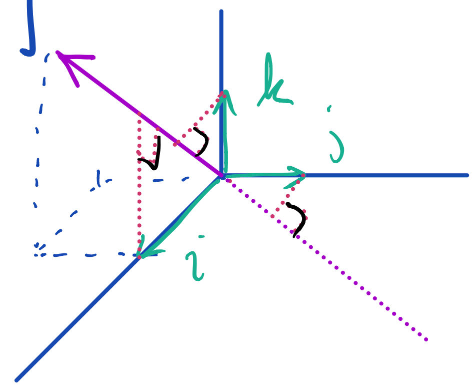




**Key:** $\vec w$ & $\vec s$ are the <u>only</u> vectors with these properties.

$$\boxed{\operatorname{proj}_{\vec v}\vec u = \frac{\vec u\cdot\vec v}{\|\vec v\|^2}\,\vec v} \qquad \boxed{\operatorname{comp}_{\vec v}\vec u = \frac{\vec u\cdot\vec v}{\|\vec v\|}}$$

<u>Example:</u> Compute the projections of $\vec i, \vec j, \vec k$ along $\begin{bmatrix}1\\-1\\2\end{bmatrix}$




$$\left\|\begin{bmatrix}1\\-1\\2\end{bmatrix}\right\|^2 = 1+1+4 = 6$$

- $$\operatorname{proj}_{\begin{bmatrix}1\\-1\\2\end{bmatrix}} i = \frac{\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\2\end{bmatrix}}{\left\|\begin{bmatrix}1\\-1\\2\end{bmatrix}\right\|^2}\begin{bmatrix}1\\-1\\2\end{bmatrix} = \frac{1}{6}\begin{bmatrix}1\\-1\\2\end{bmatrix}$$

- $$\operatorname{proj}_{\begin{bmatrix}1\\-1\\2\end{bmatrix}} j = \frac{\begin{bmatrix}0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\2\end{bmatrix}}{\left\|\begin{bmatrix}1\\-1\\2\end{bmatrix}\right\|^2}\begin{bmatrix}1\\-1\\2\end{bmatrix} = \frac{-1}{6}\begin{bmatrix}1\\-1\\2\end{bmatrix}$$

- $$\operatorname{proj}_{\begin{bmatrix}1\\-1\\2\end{bmatrix}} k = \frac{\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\2\end{bmatrix}}{\left\|\begin{bmatrix}1\\-1\\2\end{bmatrix}\right\|^2}\cdot\begin{bmatrix}1\\-1\\2\end{bmatrix} = \frac{2}{6}\begin{bmatrix}1\\-1\\2\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1\\-1\\2\end{bmatrix}$$

- $$\operatorname{comp}_{\begin{bmatrix}1\\-1\\2\end{bmatrix}} i = \frac{\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\2\end{bmatrix}}{\left\|\begin{bmatrix}1\\-1\\2\end{bmatrix}\right\|} = \frac{1}{\sqrt6}\ ;\quad \operatorname{comp}_{\begin{bmatrix}1\\-1\\2\end{bmatrix}} j = \frac{-1}{\sqrt6}\ ;\quad \operatorname{comp}_{\begin{bmatrix}1\\-1\\2\end{bmatrix}} k = \frac{2}{\sqrt6}$$

# Cross Product (in $\mathbb{R}^3$)

Fix $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$. The <u>cross product</u> $\vec{u} \times \vec{v}$ is a vector in $\mathbb{R}^3$ with coordinates:

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ -(u_1 v_3 - u_3 v_1) \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

<u>Recall</u>: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Each coord is a $2 \times 2$ determinant

<u>Q</u>: determinantal formula for $\vec{u} \times \vec{v}$?

$$\rightsquigarrow \det \begin{bmatrix} x & y & z \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} = x \det \underbrace{\begin{bmatrix} u_2 & u_3 \\ v_2 & v_3 \end{bmatrix}}_{\text{remove row 1 \& col 1}} - y \det \underbrace{\begin{bmatrix} u_1 & u_3 \\ v_1 & v_3 \end{bmatrix}}_{\text{remove row 1, col 2}} + z \det \underbrace{\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}}_{\text{remove row 1 \& col 3}}$$

So

$$\boxed{\vec{u} \times \vec{v} = \begin{bmatrix} \check{i} & \check{j} & \check{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} = \underbrace{\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}}_{=\text{scalar}} \check{i} - \underbrace{\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}}_{=\text{scalar}} \check{j} + \underbrace{\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}}_{=\text{scalar}} \check{k}}$$

$$\boxed{\vec{u}\times\vec{v} = \begin{bmatrix} \breve{i} & \breve{j} & \breve{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} = \underbrace{\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}}_{=\text{scalar}}\breve{i} - \underbrace{\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}}_{=\text{scalar}}\breve{j} + \underbrace{\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}}_{=\text{scalar}}\breve{k}}$$

**Example**: $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ $\vec{u}\times\vec{v} = \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix}$ since

$$\det\begin{bmatrix} i & j & k \\ 1 & 2 & 3 \\ 2 & -1 & 2 \end{bmatrix} = \breve{i}(2\cdot 2+1\cdot 3) - (1\cdot 2-2\cdot 3)\breve{j} + (1(-1)-2^2)\breve{k} = \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix}$$

**Properties**: $\vec{u}, \vec{v}, \vec{w}$ in $\mathbb{R}^3$, $\alpha, \beta$ scalars

1. $\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$ [ANTICOMMUTATIVE], so $\vec{u}\times\vec{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ for all $\vec{u}$
2. $\alpha\vec{u}\times\beta\vec{v} = (\alpha\beta)(\vec{u}\times\vec{v})$ [Associative], so $\vec{0}\times\vec{u} = \vec{0}$
3. $\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v} + \vec{u}\times\vec{w}$
4. $(\vec{u}+\vec{v})\times\vec{w} = \vec{u}\times\vec{w} + \vec{v}\times\vec{w}$

$\Big\}$ [Distributive]

5. $\vec{u}\cdot(\vec{v}\times\vec{w}) = (u\times\vec{v})\cdot\vec{w}$

Q: Why anticommutative? $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$.

$\det \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = -\det \begin{vmatrix} c & d \\ a & b \end{vmatrix}$ so

$$\vec{u} \times \vec{v} = \det \begin{bmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} = -\det \begin{bmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{bmatrix} = -\vec{v} \times \vec{u}$$

Key Proposition: $\vec{u} \times \vec{v} \perp \vec{u}$ & $\vec{u} \times \vec{v} \perp \vec{v}$

Why? $\vec{u} \cdot (\vec{u} \times \vec{v}) = (\underbrace{\vec{u} \times \vec{u}}_{=\vec{0}}) \cdot \vec{v} = \vec{0} \cdot \vec{v} = 0$

$$\vec{v} \cdot (\vec{u} \times \vec{v}) = -\vec{v} \, (\vec{v} \times \vec{u}) = -(\underbrace{\vec{v} \times \vec{v}}_{=\vec{0}}) \cdot \vec{u} = 0$$