Lecture 12: §2.3 Cross Product, §2.4 Lines & planes

Recall: Given $\vec{u}, \vec{v}$ in $\mathbb{R}^3$ we defined $\vec{u} \times \vec{v}$ in $\mathbb{R}^3$ via

$$\vec{u} \times \vec{v} = \begin{vmatrix} \breve{i} & \breve{j} & \breve{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} =$$

Example:

Properties: $\vec{u}, \vec{v}, \vec{w}$ in $\mathbb{R}^3$, α, β scalars

1. $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$ [ANTICOMMUTATIVE], so $\vec{u} \times \vec{u} = \vec{0}$ for all $\vec{u}$
2. $\alpha \vec{u} \times \beta \vec{v} = (\alpha\beta)(\vec{u} \times \vec{v})$ [Associative], so $\vec{0} \times \vec{u} = \vec{0}$ (take $\alpha = 0$)
3. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
4. $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$

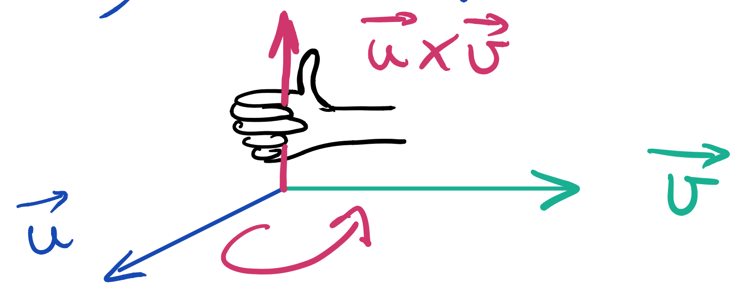
   (3 and 4: [Distributive])
5. $\vec{u} \cdot (\vec{v} \times \vec{w}) = (u \times \vec{v}) \cdot \vec{w}$

KEY PROPERTY: $\vec{u} \perp \vec{u} \times \vec{v}$ & $\vec{v} \perp \vec{u} \times \vec{v}$

Q: Direction of $\vec{u} \times \vec{v}$?

A: ① If $\vec{u}$ & $\vec{v}$ are parallel, then

② ______________ NOT parallel, direction of $\vec{u} \times \vec{v}$ is fixed by the right-hand rule



⤳ Only missing ingredient = $\|\vec{u} \times \vec{v}\|$

(*) $\boxed{\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\,\|\vec{v}\| \sin\theta}$

Lines in $\mathbb{R}^2$ & $\mathbb{R}^3$

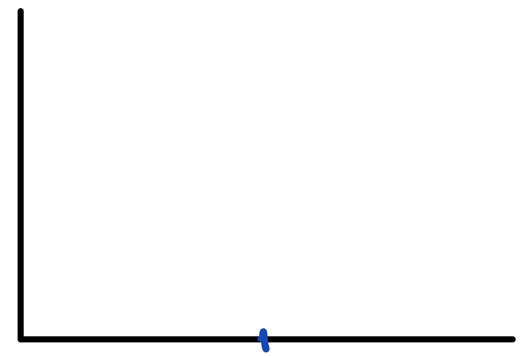
• **Know**: 2 different points in $\mathbb{R}^2$ or $\mathbb{R}^3$ determine a unique line

$\boxed{\mathbb{R}^2}$

$P_0=(x_0,y_0)$

$P_1=(x_1,y_1)$

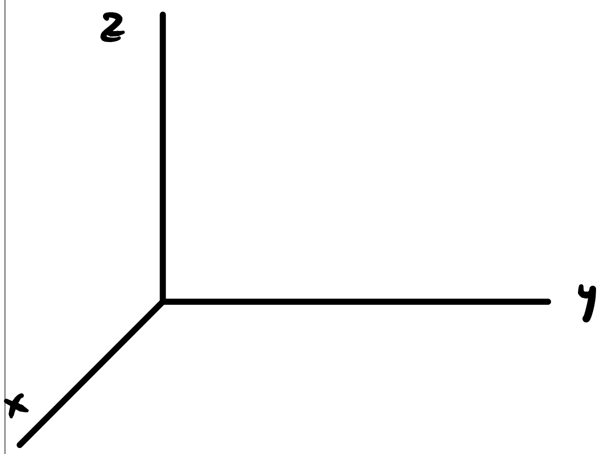
$\boxed{x_0 \neq x_1}$



$\boxed{x_0 = x_1}$

$\mathbb{R}^2$: $\overrightarrow{OP} = \overrightarrow{OP_0} + t\,\overrightarrow{P_0P_1}$ for t in $\mathbb{R}$

$\boxed{\mathbb{R}^3}$ Use the same idea!



$P_0 = (x_0, y_0, z_0)$

$P_1 = (x_1, y_1, z_1)$

$P = (x, y, z)$

Eqn for a line in $\mathbb{R}^3$:

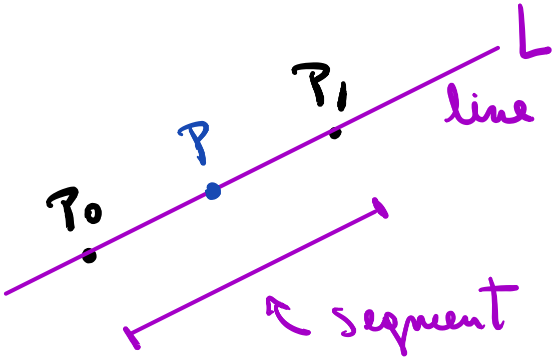
$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$$

for some t.

Example (1) Find the equation of the line L which is parallel to $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ & passes through $(5,0,2)$

(2) Find the intersection of the line L with the xy-plane

Line Segments

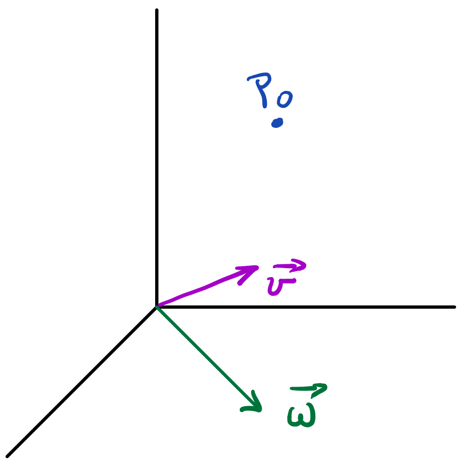


We are only interested in the points along the line L lying in between P_0 & P_1

Planes in 3-Space

2 ways

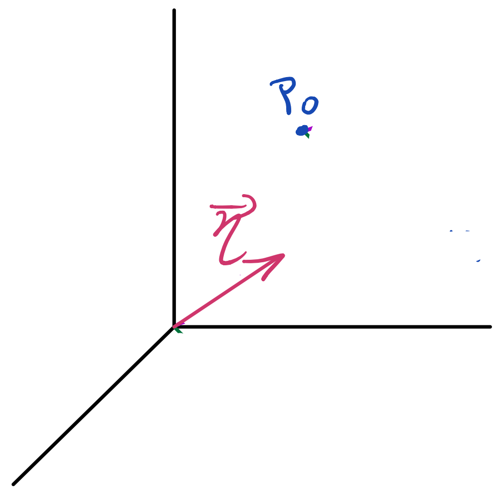
(1) A point P_0 & 2 non-parallel directions $\vec{v}$ & $\vec{w}$



[Equivalently: 3 non collinear pts P_0, Q_0, R_0]

Take: $\vec{v} = \overrightarrow{P_0Q_0}$ & $\vec{w} = \overrightarrow{P_0R_0}$

(2) A point P_0 & a normal $\vec{n}$



$$\boxed{a(x-x_0)+b(y-y_0)+c(z-z_0)=0}$$

$\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $P_0=(x_0,y_0,z_0)$

Example: Find the equation of the plane passing through $P_0=(1,0,0)$
$Q_0=(2,1,-1)$
$R_0=(1,1,1)$