Lecture 12: §2.3 Cross Product, §2.4 Lines & planes

Recall: Given $\vec{u}, \vec{v}$ in $\mathbb{R}^3$ we defined $\vec{u} \times \vec{v}$ in $\mathbb{R}^3$ via

det!

$$\boxed{\vec{u} \times \vec{v} = \begin{vmatrix} \check{i} & \check{j} & \check{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \underbrace{\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}}_{=\text{scalar}} \check{i} - \underbrace{\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}}_{=\text{scalar}} \check{j} + \underbrace{\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}}_{=\text{Scalar}} \check{k} = \begin{bmatrix} u_2v_3 - u_3v_3 \\ -(u_1v_3 - u_3v_1) \\ u_1v_2 - u_2v_1 \end{bmatrix}}$$

Example: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{vmatrix} \check{i} & \check{j} & \check{k} \\ 1 & 2 & 3 \\ 2 & -1 & 2 \end{vmatrix} = \check{i}(4+3) - \check{j}(2-6) + (-1-4)\check{k} = \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix}$

Properties: $\vec{u}, \vec{v}, \vec{w}$ in $\mathbb{R}^3$, α, β scalars

1. $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$ [ANTICOMMUTATIVE], so $\vec{u} \times \vec{u} = \vec{0}$ for all $\vec{u}$
2. $\alpha \vec{u} \times \beta \vec{v} = (\alpha\beta)(\vec{u} \times \vec{v})$ [Associative], so $\vec{0} \times \vec{u} = \vec{0}$ (take $\alpha = 0$)
3. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
4. $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$

[Distributive] (for 3 and 4)

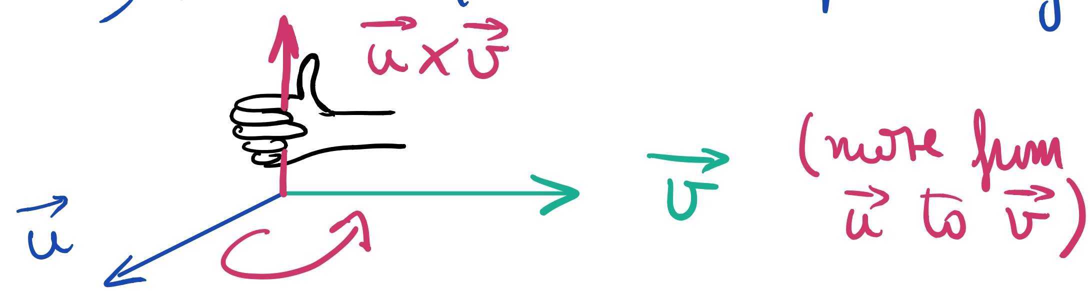
5. $\vec{u} \cdot (\vec{v} \times \vec{w}) = (u \times \vec{v}) \cdot \vec{w}$

KEY PROPERTY: $\vec{u} \perp \vec{u}\times\vec{v}$ & $\vec{v} \perp \vec{u}\times\vec{v}$

Q: Direction of $\vec{u}\times\vec{v}$?

A: ① If $\vec{u}$ & $\vec{v}$ are parallel [θ between them $= 0$ or $180°$], then $\vec{u}\times\vec{v} = \vec{0}$ (so no direction!)

② ________ NOT parallel, direction of $\vec{u}\times\vec{v}$ is fixed by the right-hand rule



⇝ Only missing ingredient $= \|\vec{u}\times\vec{v}\|$

Geometry of Cross Products: If $\vec{u}, \vec{v} \neq \vec{0}$, write $\theta =$ angle between them. ($0 \le \theta \le 180°$)

Then:

$$\boxed{\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\,\|\vec{v}\|\sin\theta}$$

(The θ can be recovered from $\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$)

$(*)$ $$\boxed{\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\,\|\vec{v}\|\sin\theta}$$

$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

Why? Claim $\|\vec{u}\times\vec{v}\|^2 + (\vec{u}\cdot\vec{v})^2 = \|\vec{u}\|^2\|\vec{v}\|^2$

Why? $\|\vec{u}\times\vec{v}\|^2 = \left\| \begin{bmatrix} u_2v_3 - u_3v_1 \\ -(u_1v_3 - u_3v_1) \\ u_1v_2 - u_2v_1 \end{bmatrix} \right\|^2 = (u_2v_3 - u_3v_1)^2 + (u_1v_3 - u_3v_1)^2 + (u_1v_2 - u_2v_1)^2$

$+$

$(\vec{u}\cdot\vec{v})^2 = (u_1v_1 + u_2v_2 + u_3v_3)^2$

$\|\vec{u}\times\vec{v}\|^2 + (\vec{u}\cdot\vec{v})^2 = (u_2v_3)^2 + (u_3v_1)^2 + (u_1v_3)^2 + (u_3v_1)^2 + (u_1v_2)^2 + (u_2v_1)^2$
$-2(u_2u_3v_1v_3 + u_1u_3v_1v_3 + u_1u_2v_1v_2) + (u_1v_1)^2 + (u_2v_2)^2 + (u_3v_3)^2 +$
$+2(u_2u_3v_1v_3 + u_1u_3v_1v_3 + u_1u_2v_1v_2) = (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2)$
$= \|\vec{u}\|^2\|\vec{v}\|^2$

So $\|\vec{u}\times\vec{v}\|^2 = \|\vec{u}\|^2\|\vec{v}\|^2 - (u\cdot v)^2$
$= \|\vec{u}\|^2\|\vec{v}\|^2 - (\|\vec{u}\|\,\|\vec{v}\|\cos\theta)^2$
$= \|\vec{u}\|^2\|\vec{v}\|^2(1 - \cos^2\theta) = \|\vec{u}\|^2\|\vec{v}\|^2\sin^2\theta$

But $\sin\theta \geq 0$ if $0 \leq \theta \leq 180^\circ$, so take $\sqrt{\ }$ on both sides to get $(*)$

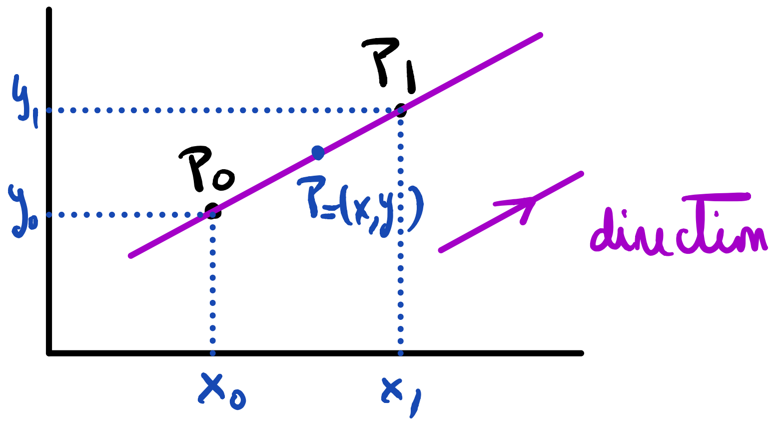
Lines in $\mathbb{R}^2$ & $\mathbb{R}^3$

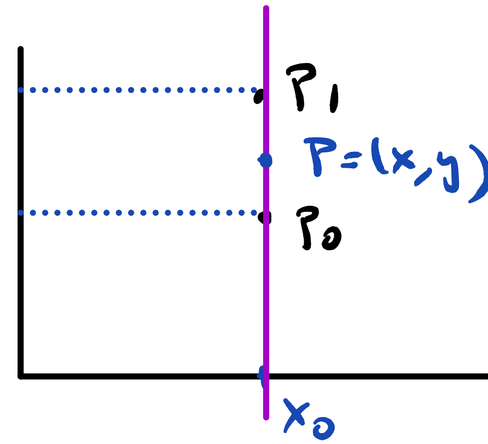
- **Know:** 2 different points in $\mathbb{R}^2$ or $\mathbb{R}^3$ determine a unique line

$\mathbb{R}^2$

$P_0 = (x_0, y_0)$

$P_1 = (x_1, y_1)$





$\boxed{x_0 \neq x_1}$

$\boxed{x_0 = x_1}$

Eqn. $y = m(x - x_0) + y_0$

with $m = \dfrac{y_1 - y_0}{x_1 - x_0}$ = slope

Eqn: $x = x_0$

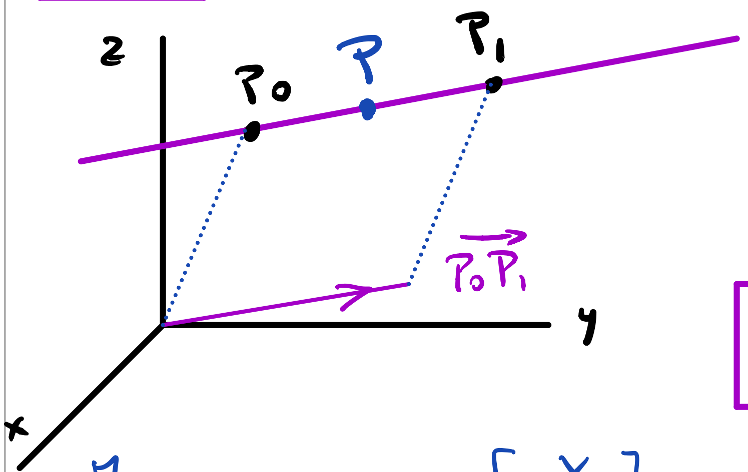
Parametric equation: direction $= \overrightarrow{P_0P_1} = \begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \end{bmatrix}$

t = parameter

$$\leadsto \boxed{\overrightarrow{OP} = \overrightarrow{OP_0} + t\,\overrightarrow{P_0P_1} \quad \text{for } t \text{ in } \mathbb{R}}$$

$\mathbb{R}^2$: $\overrightarrow{OP} = \overrightarrow{OP_0} + t\,\overrightarrow{P_0P_1}$ for t in $\mathbb{R}$

$\boxed{\mathbb{R}^3}$ Use the same idea!



$P_0 = (x_0, y_0, z_0)$

$P_1 = (x_1, y_1, z_1)$

$P = (x, y, z)$

$\boxed{\underline{\text{Vector Eqn}}: \ \overrightarrow{OP} = \overrightarrow{OP_0} + t\,\overrightarrow{P_0P_1} \quad \text{for } t \text{ in } \mathbb{R}}$

Meaning:
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \\ z_1 - z_0 \end{bmatrix}$$
for some choice of t

given

our unknown.

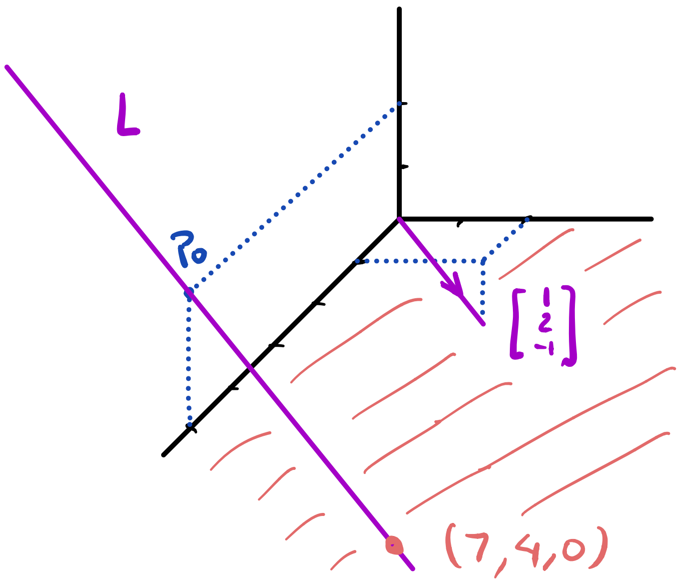
Write direction vector $= \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

point P_0 direction vector

• Each component gives one condition:
$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$$
for some t.

Eqn for a line in $\mathbb{R}^3$:
$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$$ for some t.

Example (1) Find the equation of the line L which is parallel to $\begin{bmatrix}1\\2\\-1\end{bmatrix}$ & passes through $(5,0,2)$



$P_0 = (5,0,2)$ $\quad \vec{v} = \begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}1\\2\\-1\end{bmatrix}$

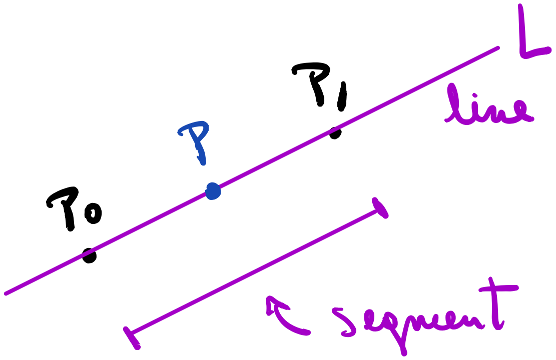
A $\begin{cases} x = 5 + t \\ y = 0 + 2t \\ z = 2 + (-1)t \end{cases}$ $\quad (*)$

(2) Find the intersection of the line L with the XY-plane

A XY-plane has equation $z=0$. So we get one equation in t: $\quad 0 = 2 - t \rightsquigarrow \boxed{t=2}$ & Substitute in $(*)$:

$\rightsquigarrow$ Pt of intersection $= (5+2,\ 0+2\cdot 2,\ 0) = (7,4,0)$

Line Segments

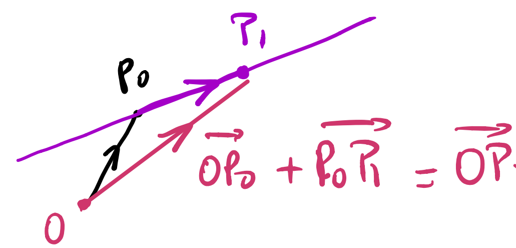


We are only interested in the points along the line L lying in between P_0 & P_1

$\underline{L}$: $\overrightarrow{OP} = \overrightarrow{OP_0} + t\,\overrightarrow{P_0P_1}$ for t in $\mathbb{R}$

For the segment, we restrict the range for t

- If $P = P_0$: $\overrightarrow{OP_0} = \overrightarrow{OP_0} + 0\,\overrightarrow{P_0P_1}$
- If $P = P_1$: $\overrightarrow{OP_1} = \overrightarrow{OP_0} + 1\,\overrightarrow{P_0P_1}$



$$\leadsto \boxed{\underline{\text{Segment}}: \quad \overrightarrow{P_0P} = t\,\overrightarrow{P_0P_1} \quad \text{for} \quad 0 \le t \le 1}$$

Special case: Midpoint P between P_0 & P_1 comes from $\overrightarrow{P_0P} = \frac{1}{2}\overrightarrow{P_0P_1}$

$P_0 = (x_0, y_0, z_0)$

$P_1 = (x_1, y_1, z_1)$

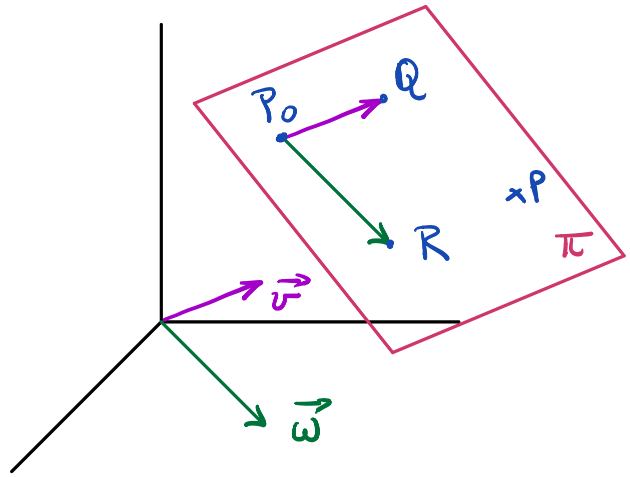
$P = (x, y, z)$

$$\leadsto \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \\ z_1 - y_0 \end{bmatrix} \quad \text{gives} \quad \boxed{\begin{aligned} x &= \frac{x_0 + x_1}{2} \\ y &= \frac{y_0 + y_1}{2} \\ z &= \frac{z_0 + z_1}{2} \end{aligned}}$$

Planes in 3-Space

2 ways

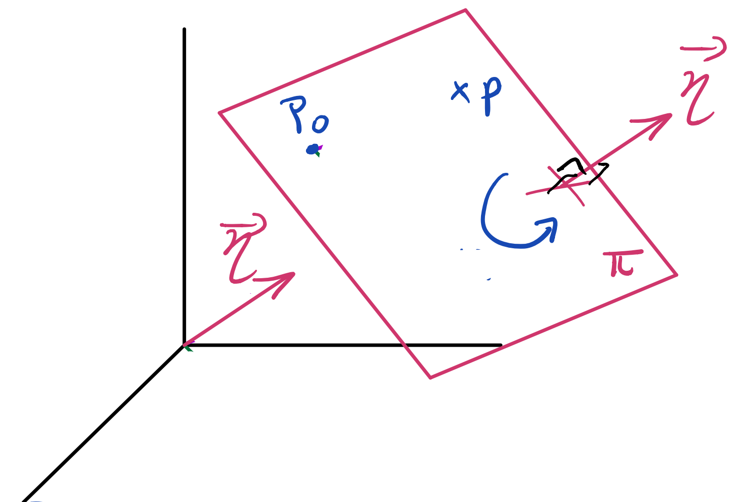
① A point P_0 & 2 non-parallel directions $\vec{v}$ & $\vec{w}$



[Equivalently: 3 non collinear pts P_0, Q_0, R_0]

Take: $\vec{v} = \overrightarrow{P_0Q_0}$ & $\vec{w} = \overrightarrow{P_0R_0}$

② A point P_0 & a normal $\vec{n}$



- $\vec{n}$ orients the plane (right hand rule)
- $\vec{n} \perp \vec{v}$ & $\vec{n} \perp \vec{w}$

$\rightsquigarrow$ Take $\boxed{\vec{n} = \pm \vec{v} \times \vec{w}}$ (both work!)

Vector equation: $\boxed{\overrightarrow{P_0P} \cdot \vec{n} = 0}$

Explicit: $P = (x, y, z)$

$P_0 = (x_0, y_0, z_0)$

$\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \rightsquigarrow \boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$

$$\boxed{a(x-x_0)+b(y-y_0)+c(z-z_0)=0}$$

$\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $P_0=(x_0,y_0,z_0)$

Example: Find the equation of the plane passing through $P_0=(1,0,0)$, $Q_0=(2,1,-1)$, $R_0=(1,1,1)$

First, we find 2 directions $\vec{v} = \overrightarrow{P_0Q_0} = \begin{bmatrix} 2-1 \\ 1-0 \\ -1-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$\vec{w} = \overrightarrow{P_0R_0} = \begin{bmatrix} 1-1 \\ 1-0 \\ 1-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\rightsquigarrow$ Normal: $\vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \vec{k}$

$= (1+1)\,\vec{i} - \vec{j} + \vec{k} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Equation: $2(x-1)+(-1)(y-0)+1(z-0)=0$

$2x-y+z-2=0$, or $\boxed{2x-y+z=2}$ $(*)$

Easy check: P_0, Q_0, R_0 satisfy $(*)$

P_0: $2\cdot 1-0+0=2$ ✓ ; Q_0: $2.2-1+(-1)=2$ ✓ R_0: $2.1-1+1=2$ ✓