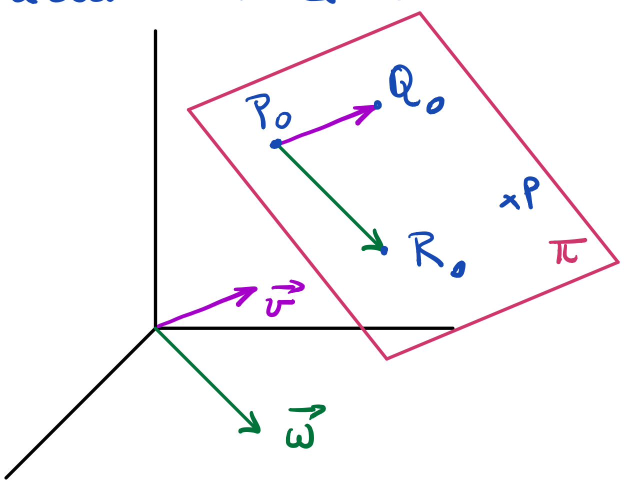
Lecture 13 §2.4 Planes in 3-Space

Recall: Two ways to describe planes in 3-Space

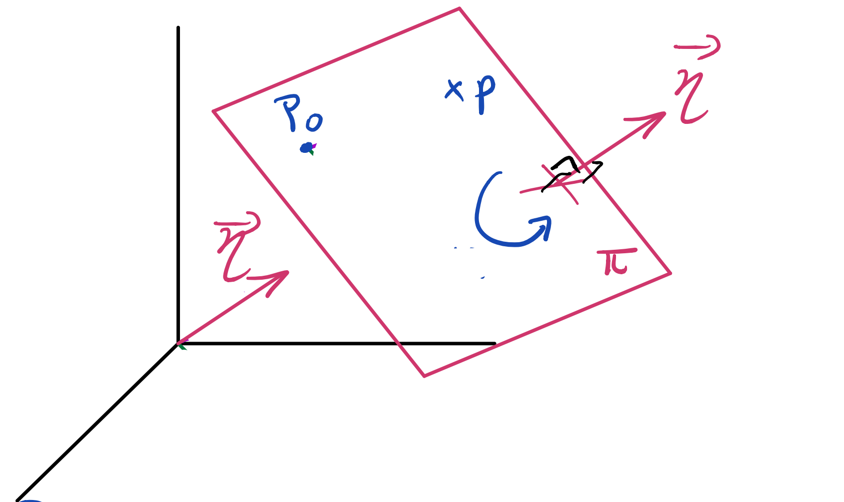
① A point P_0 & 2 non-parallel directions $\vec{v}$ & $\vec{w}$



[Equivalently: 3 non collinear pts P_0, Q_0, R_0]

Take: $\vec{v} = \overrightarrow{P_0Q_0}$ & $\vec{w} = \overrightarrow{P_0R_0}$

② A point P_0 & a normal $\vec{n}$



- $\vec{n}$ orients the plane (right hand rule)
- $\vec{n} \perp \vec{v}$ & $\vec{n} \perp \vec{w}$

$\rightsquigarrow$ Take $\boxed{\vec{n} = \vec{v} \times \vec{w}}$

Explicit: $P = (x, y, z)$

$P_0 = (x_0, y_0, z_0)$

$\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \rightsquigarrow \boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$

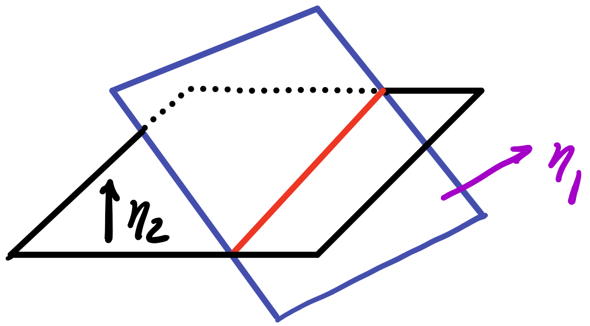
<u>Example 1</u>: Compute the intersection of the plane $2x - y + z = 2$ with the 3 coordinate planes

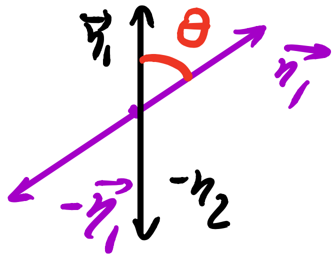
Example 2 Find the intersection of the plane $2x - y + z = 2$ with the line with direction $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ passing through $(3, 4, 5)$.

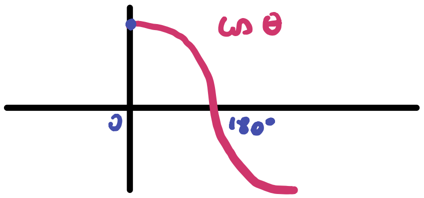
Example 3: Same question but with line L' through $(3, 4, 5)$ with direction $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

Parallel & Orthogonal Planes

Definition: The angle between 2 planes is the acute angle between the normal directions







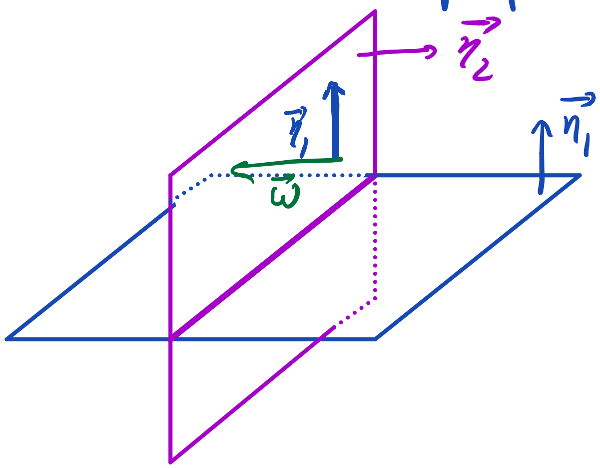
In particular: $\theta = 0$ gives parallel planes $(\vec{n}_1 \parallel \vec{n}_2)$

$\theta = 90°$ gives perpendicular or orthogonal planes $(\vec{n}_1 \perp \vec{n}_2)$

Example 1: Find the plane parallel to $3x - 2y + 5z = 4$ passing through $(1,-1,1)$

Obs: 1. Parallel planes have parallel normals (proportional)

2. What about perpendicular planes?



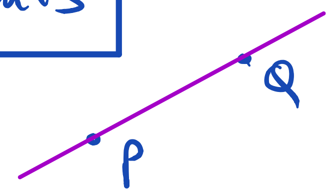
$\vec{n}_2 \perp \vec{n}_1$ means $\vec{n}_1$ is one of the directions in the plane with normal $\vec{n}_2$

$\rightsquigarrow$ The 2^{nd} direction is given by $\vec{\omega}$ with $\vec{\omega} \perp \vec{n}_2$ & $\vec{\omega} \not\parallel \vec{n}_1$

Example 2: Find a plane orthogonal to $3x-2y+5z=4$, passing through $P_0=(1,0,0)$ & $Q_0=(1,1,0)$.

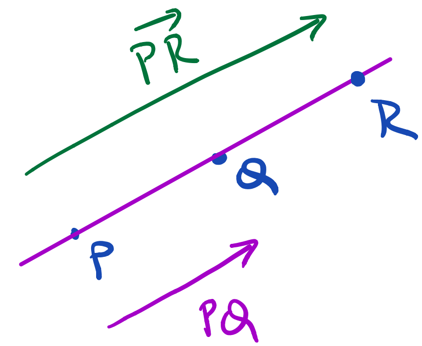
Collinear Points

- 2 points are always collinear



- 3 points are in general not collinear

Condition for being collinear: $P = (p_1, p_2, p_3)$
$Q = (q_1, q_2, q_3)$
$R = (r_1, r_2, r_3)$



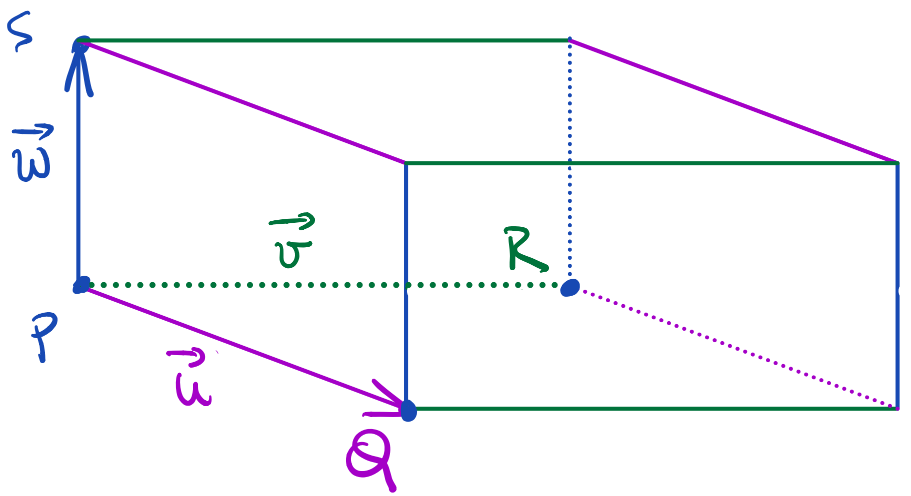
$$\boxed{\overrightarrow{PQ} \text{ is proportional to } \overrightarrow{PR} \text{ if and only if } P, Q, R \text{ are collinear}}$$

This means we can find t in $\mathbb{R}$ with $\begin{bmatrix} q_1 - p_1 \\ q_2 - p_2 \\ q_3 - p_3 \end{bmatrix} = t \begin{bmatrix} r_1 - p_1 \\ r_2 - p_2 \\ r_3 - p_3 \end{bmatrix}$

<u>Example</u>: Check if $(1,0,1)$, $(2,3,4)$ & $(6,7,8)$ are collinear or not

Coplanar Points

- 3 distinct & not collinear points always determine a unique plane
- 4 points in general do <u>not</u> lie in the same plane (they are not coplanar)



Here: P, Q, S determine a unique plane, but R is outside the plane

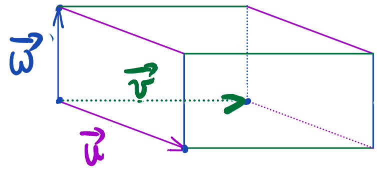
⇝ Parallelepiped is NOT flat!

<u>Rule</u>:

$$\boxed{P, Q, R, S \text{ are coplanar if and only if parallelepiped is flat}}$$

Flat means volume = 0.

$$\boxed{\text{Volume} = |\vec{u} \cdot (\vec{v} \times \vec{w})| = |(\vec{u} \times \vec{v}) \cdot \vec{w}| = |(\vec{u} \times \vec{w}) \cdot \vec{v}|.}$$

$\rightsquigarrow$ Volume (

) $= |\vec{u} \cdot (\vec{v} \times \vec{w})|$

<u>Example</u>: Show that $\underset{P}{(1,3,2)}$, $\underset{Q}{(3,-1,6)}$, $\underset{R}{(5,2,0)}$ & $\underset{S}{(3,6,-4)}$ are coplanar.