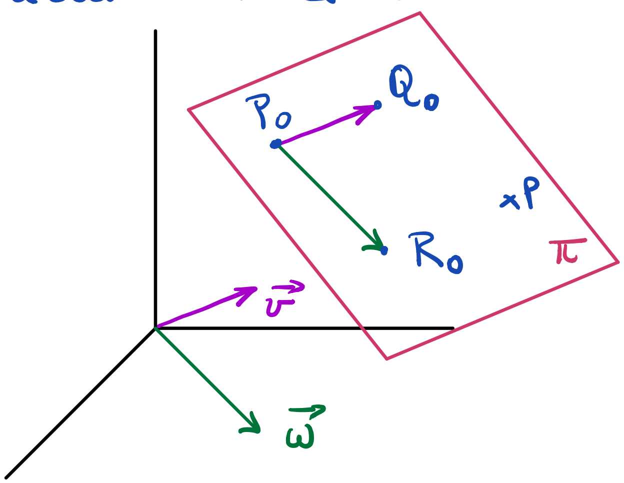
Lecture 13 §2.4 Planes in 3-Space

Recall: Two ways to describe planes in 3-Space

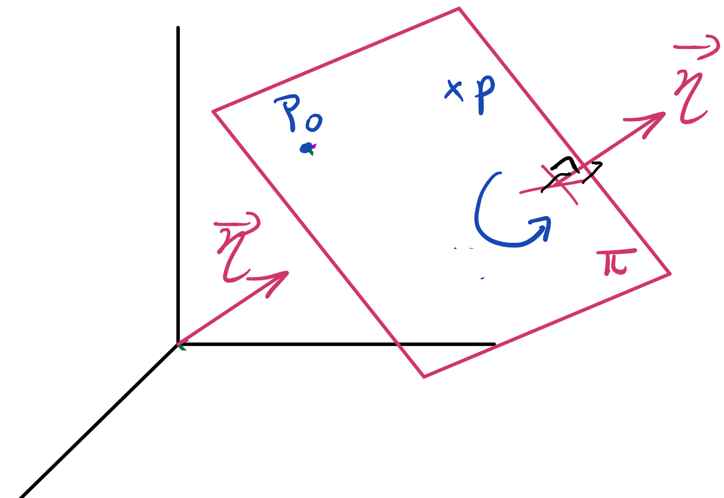
① A point P_0 & 2 non-parallel directions $\vec{v}$ & $\vec{w}$



[**Equivalently**: 3 non collinear pts P_0, Q_0, R_0]

Take: $\vec{v} = \overrightarrow{P_0Q_0}$ & $\vec{w} = \overrightarrow{P_0R_0}$

② A point P_0 & a normal $\vec{n}$



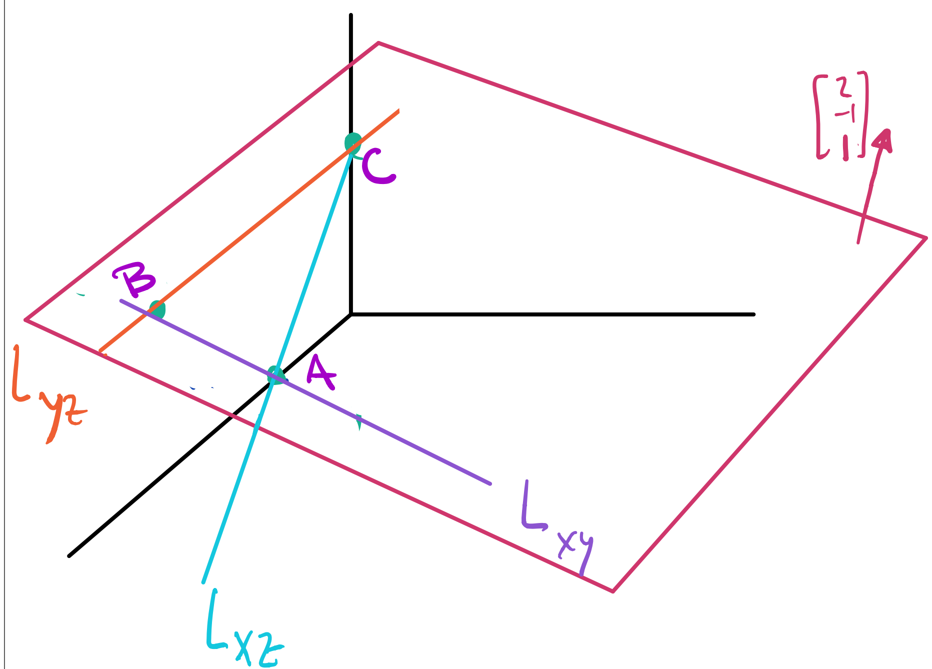
- $\vec{n}$ orients the plane (right hand rule)
- $\vec{n} \perp \vec{v}$ & $\vec{n} \perp \vec{w}$

$\rightsquigarrow$ Take $\boxed{\vec{n} = \vec{v} \times \vec{w}}$

Explicit: $P = (x, y, z)$
$P_0 = (x_0, y_0, z_0)$

$$\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \rightsquigarrow \boxed{a(x-x_0)+b(y-y_0)+c(z-z_0)=0}$$

Example 1: Compute the intersection of the plane $2x - y + z = 2$ with the 3 coordinate planes



xy-plane: $\begin{cases} 2x - y + z = 2 \\ z = 0 \end{cases}$

$\rightsquigarrow$ Line: L_{xy}: $2x - y = 2$ in $(z=0)$

yz-plane $\begin{cases} 2x - y + z = 2 \\ x = 0 \end{cases}$

$\rightsquigarrow$ Line L_{yz} $-y + z = 2$ in $(x=0)$

xz-plane $\begin{cases} 2x - y + z = 2 \\ y = 0 \end{cases}$

$\rightsquigarrow$ Line L_{xz}: $2x + z = 2$ in $(y=0)$

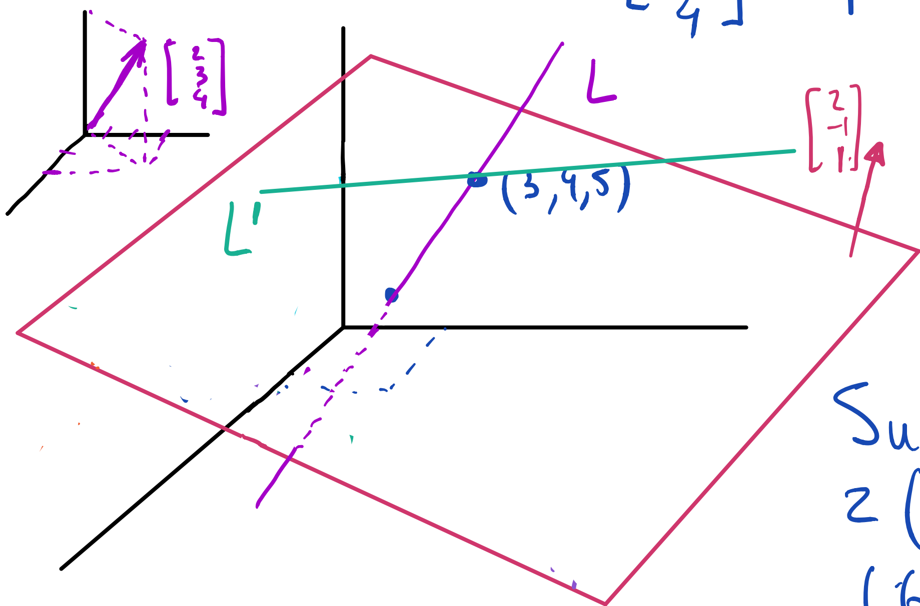
Draw intersections with 3 axes: $A = (1,0,0)$, $B = (0,-2,0)$, $C = (0,0,2)$

x-axis $(x,0,0)$ in $2x - y + z = 2$ gives $x = 1$ $(2x - 0 + 0 = 2)$

y-axis $(0,y,0)$ ———————— $y = -2$ $(2 \cdot 0 - y + 0 = 2)$

z-axis $(0,0,z)$ ———————— $z = 2$ $(2 \cdot 0 - 0 + z = 2)$

Example 2 Find the intersection of the plane $2x - y + z = 2$ with the line with direction $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ passing through $(3,4,5)$.



Line equations:

$$x = 3 + t\,2$$
$$y = 4 + t\,3 \qquad \text{for } t \text{ in } \mathbb{R}$$
$$z = 5 + t\,4$$

Substitute these in the plane's equation:

$$2(3+2t) - (4+3t) + (5+4t) = 2$$
$$(6-4+5) + (4-3+4)\,t = 2$$
$$7 + 5t = 2 \rightsquigarrow \boxed{t=-1}$$

Conclusion: $(x,y,z) = (3-2,\, 4-3,\, 5-4) = (1,1,1)$

Example 3: Same question but with line L' through $(3,4,5)$ with direction $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

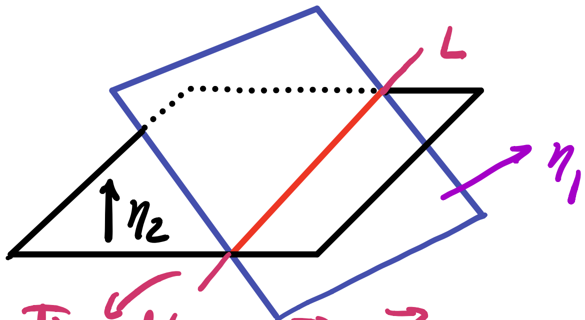
$$\begin{cases} x = 3-t \\ y = 4-t \\ z = 5+t \end{cases} \rightsquigarrow \begin{array}{l} 2(3-t) - (4-t) + (5+t) = 2 \\ (6-4+5) + 0t = 2 \\ \quad 7 = 2 \end{array}$$

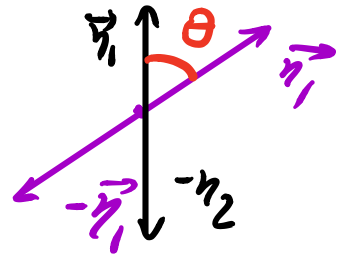
No solution!

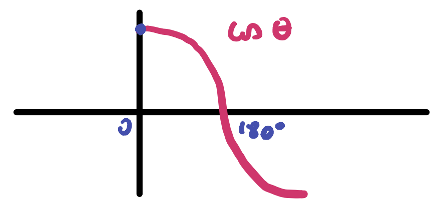
Conclude: No intersection.

Parallel & Orthogonal Planes

<u>Definition</u>: The angle between 2 planes is the acute angle between the normal directions







direction of L: $\vec{\eta}_1 \times \vec{\eta}_2$

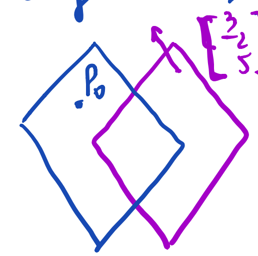
Use $|\vec{\eta}_1 \cdot \vec{\eta}_2| = \|\vec{\eta}_1\| \, \|\vec{\eta}_2\| \, |\cos\theta|$ to find $0 \le \theta \le 90^\circ$

<u>In particular</u>: $\theta = 0$ gives parallel planes $(\vec{\eta}_1 \parallel \vec{\eta}_2)$

$\theta = 90^\circ$ gives perpendicular or orthogonal planes $(\vec{\eta}_1 \perp \vec{\eta}_2)$

<u>Example 1</u>: Find the plane parallel to $3x - 2y + 5z = 4$ passing through $(1,-1,1)$

Take $\vec{\eta}_2 = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$ (or any multiple!) $P_0 = (1,-1,1)$

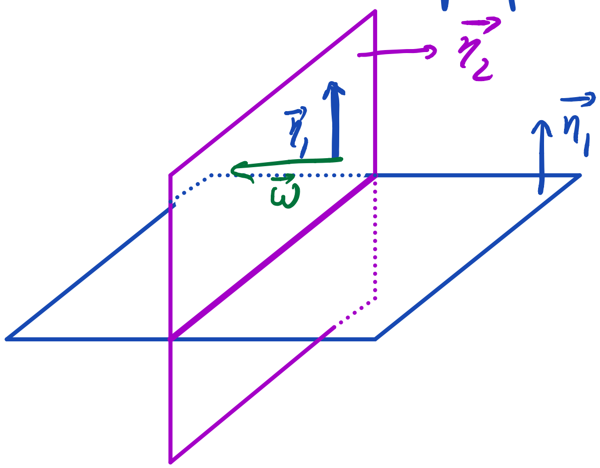


$\leadsto$ <u>Eqn</u>: $3(x-1) + (-2)(y-(-1)) + 5(z-1) = 0$

$$\boxed{3x - 2y + 5z = 10}$$

Obs: ① Parallel planes have parallel normals (proportional)

② What about perpendicular planes?



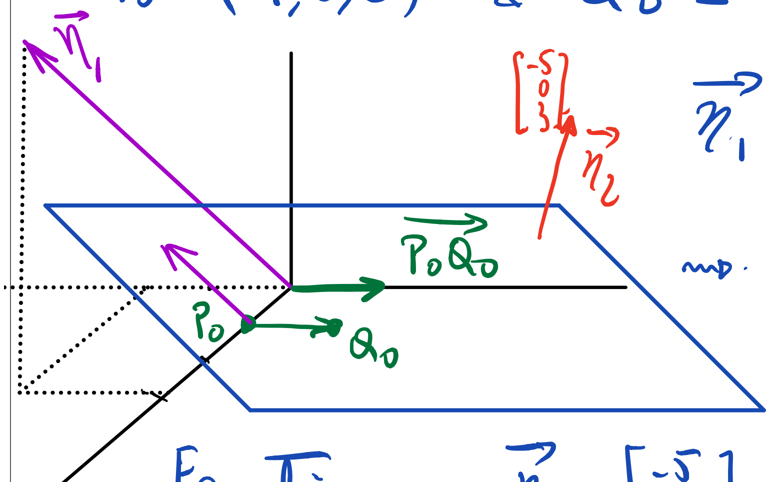
$\vec{n_2} \perp \vec{n_1}$ means $\vec{n_1}$ is one of the directions in the plane with normal $\vec{n_2}$

⇝ The 2^{nd} direction is given by $\vec{w}$ with $\vec{w} \perp \vec{n_2}$ & $\vec{w} \not\parallel \vec{n_1}$

(Eg: $\vec{w} = \vec{n_1} \times \vec{n_2}$ works)

Example 2: Find a plane orthogonal to $3x - 2y + 5z = 4$, passing through $P_0 = (1,0,0)$ & $Q_0 = (1,1,0)$.

(Check: P_0: $-5 = -5$ ✓
Q_0: $-5 + 0 = -5$ ✓)



$$\vec{n_1} = \begin{bmatrix}3\\-2\\5\end{bmatrix} \quad \& \quad 2^{nd} \text{ direction is } \overrightarrow{P_0Q_0} = \begin{bmatrix}1-1\\1-0\\0-0\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

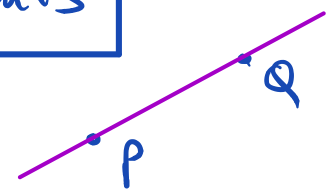
$$\rightsquigarrow \vec{n_2} = \vec{n_1} \times \overrightarrow{P_0Q_0} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & 5 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} -2 & 5 \\ 1 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} 3 & 5 \\ 0 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} \vec{k} = \begin{bmatrix}-5\\0\\3\end{bmatrix}$$

⇝ Equation uses $\vec{n_2} = \begin{bmatrix}-5\\0\\3\end{bmatrix}$ & $P_0 = (1,0,0)$ ⇝ $-5(x-1) + 0(y-0) + 3(z-0) = 0$

$$\boxed{-5x + 3z = -5}$$

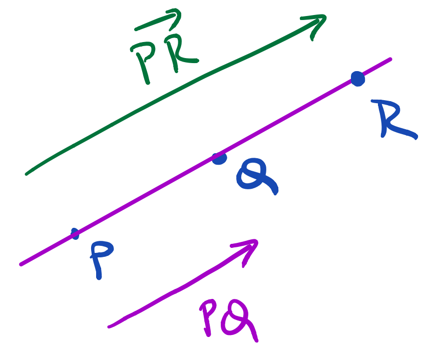
Collinear Points

- 2 points are always collinear



- 3 points are in general not collinear

Condition for being collinear: $P = (p_1, p_2, p_3)$
$Q = (q_1, q_2, q_3)$
$R = (r_1, r_2, r_3)$



$\boxed{\vec{PQ} \text{ is proportional to } \vec{PR} \text{ if and only if } P, Q, R \text{ are collinear}}$

This means we can find t in $\mathbb{R}$ with $\begin{bmatrix} q_1 - p_1 \\ q_2 - p_2 \\ q_3 - p_3 \end{bmatrix} = t \begin{bmatrix} r_1 - p_1 \\ r_2 - p_2 \\ r_3 - p_3 \end{bmatrix}$

<u>Example</u>: Check if $\underset{P}{(1,0,1)}$, $\underset{Q}{(2,3,4)}$ & $\underset{R}{(6,7,8)}$ are collinear or not

<u>A</u>: $\vec{PQ} = \begin{bmatrix} 2-1 \\ 3-0 \\ 4-1 \end{bmatrix}$, $\vec{PR} = \begin{bmatrix} 6-1 \\ 7-0 \\ 8-1 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 7 \end{bmatrix}$

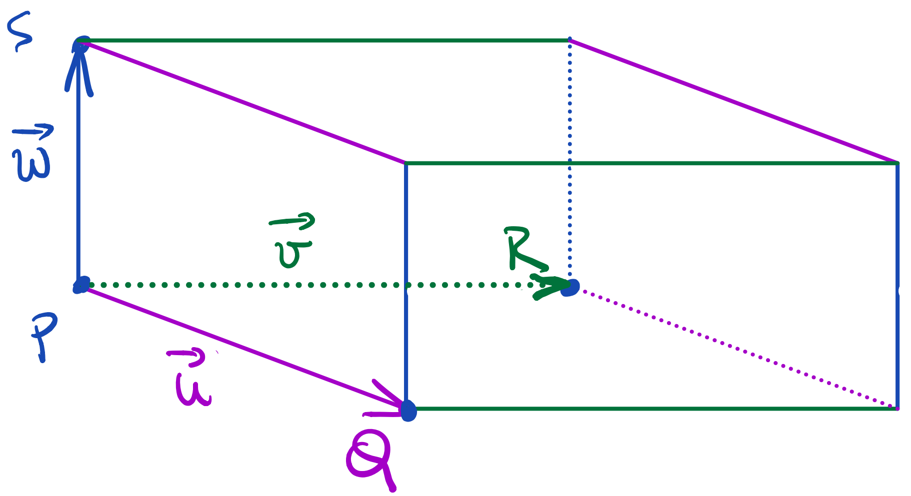
<u>Q</u>: Can we solve $\begin{cases} 1 = t5 \,? \\ 3 = t7 \\ 3 = t7 \end{cases}$ $\rightsquigarrow t = \frac{1}{5}$ $\rightsquigarrow t = \frac{3}{7}$ <u>Issue</u>

<u>A</u>: NO!

$\rightsquigarrow$ NOT collinear.

Coplanar Points

- 3 distinct & not collinear points always determine a unique plane
- 4 points in general do <u>not</u> lie in the same plane (they are not coplanar)



Here: P, Q, S determine a unique plane, but R is outside the plane

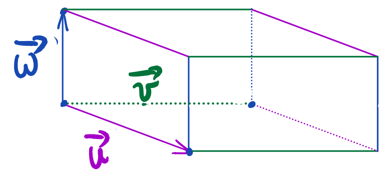
$\rightsquigarrow$ Parallelepiped is NOT flat!

<u>Rule</u>:

$\boxed{\text{P, Q, R, S are coplanar if and only if parallelepiped is flat}}$

Flat means volume $= 0$.

$$\boxed{\text{Volume} = |\vec{u} \cdot (\vec{v} \times \vec{w})| = |(\vec{u} \times \vec{v}) \cdot \vec{w}| = |(\vec{u} \times \vec{w}) \cdot \vec{v}|.}$$

$\rightsquigarrow$ Volume () $= |\vec{u} \cdot (\vec{v} \times \vec{w})|$

Example: Show that $(1,3,2)$, $(3,-1,6)$, $(5,2,0)$ & $(3,6,-4)$ are coplanar.
$\qquad\qquad P \qquad\qquad Q \qquad\qquad R \qquad\qquad S$

Solution 1: Use the volume formula

$\vec{u} = \overrightarrow{PQ} = \begin{bmatrix} 3-1 \\ -1-3 \\ 6-2 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 4 \end{bmatrix}, \quad \vec{w} = \overrightarrow{PS} = \begin{bmatrix} 5-1 \\ 2-3 \\ 0-2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix}, \quad \vec{v} = \overrightarrow{PR} = \begin{bmatrix} 3-1 \\ 6-3 \\ -4-2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix}$

$\rightsquigarrow \text{Vol} = \begin{bmatrix} 2 \\ -4 \\ 4 \end{bmatrix} \cdot \left(\begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix} \times \begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ -4 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -12 \\ -20 \\ -14 \end{bmatrix} = 2(-12) + (-4)(-20) + 4(-14)$
$= -24 + 80 - 56 = 0$

$$\begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix} \times \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = \det \begin{pmatrix} i & j & k \\ 2 & 3 & -6 \\ 4 & -1 & -2 \end{pmatrix} = \underbrace{\begin{vmatrix} 3 & -6 \\ -1 & -2 \end{vmatrix}}_{(-6-6)} i - \underbrace{\begin{vmatrix} 2 & -6 \\ 4 & -2 \end{vmatrix}}_{(-4+24)} j + \underbrace{\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix}}_{(-2-12)} k = \begin{bmatrix} -12 \\ -20 \\ -14 \end{bmatrix}$$

Solution 2 Find the equation of the plane through P, S, R & check Q is in it!

$n = \overrightarrow{PS} \times \overrightarrow{PR} = \begin{bmatrix} -12 \\ -20 \\ -14 \end{bmatrix}$
$P = (1,3,2)$

Eqn: $-12(x-1) + (-20)(y-3) + (-14)(z-2) = 0$
$12x + 20y + 14z = 100$

Check Q: $12(3) + 20(-1) + 14(6) = 100$ ✓