Lecture 14: §3.1-2: Intro & Vector Space Properties of $\mathbb{R}^n$
§3.3 Examples of Subspaces of $\mathbb{R}^n$

So far we have seen 2 constructions:

① (Column) Vectors in $\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^4, \ldots$

② Solutions to homogeneous systems in $\mathbb{R}^n$ can be written as:

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \alpha_1 \vec{V_1} + \alpha_2 \vec{V_2} + \cdots + \alpha_m \vec{V_m} \qquad (\alpha_1, \ldots \alpha_m \text{ in } \mathbb{R})$$

where $\alpha_1, \alpha_2, \ldots, \alpha_m$ are the m independent variables of $A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

$\text{rank}(A) = n - m$.

Q: What do these 2 constructions have in common?

Defining properties for $\mathbb{R}^n$

Theorem 1: Write $\mathbb{V} = \mathbb{R}^n$. For $\vec{x}, \vec{y}, \vec{z}$ in $\mathbb{V}$, α, β scalars, we have

1. Closure Properties: (C1)

   (C2)

2. Addition Properties: (A1)

   (A2)

   (A3)

   (A4)

3. Scalar Mult. Properties: (M1)

   (M2)

   (M3)

   (M4)

Subspaces of $\mathbb{R}^n$.

Def A subset $\mathbb{W}$ of $\mathbb{R}^n$ is a **subspace** if these 10 properties hold for $\mathbb{W}$

Key Fact: Given a subset $\mathbb{W}$ for $\mathbb{R}^n$, we don't need to check **all** 10 properties to see if $\mathbb{W}$ is a subspace.

(A1) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$

(A2) $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$

(M1) $\alpha(\beta \vec{X}) = (\alpha\beta)\vec{X}$

(M2) $\alpha(\vec{X} + \vec{Y}) = \alpha\vec{X} + \alpha\vec{Y}$

(M3) $(\alpha + \beta)\vec{X} = \alpha\vec{X} + \beta\vec{X}$

(M4) $1\vec{X} = \vec{X}$ for all $\vec{X}$

Theorem 2: A subset W in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if and only if

(S1) The zero vector lies in W.

(S2) $\vec{x}+\vec{y}$ lies in W whenever $\vec{x}, \vec{y}$ are in W

(S3) $\alpha \vec{x}$ ______________________ $\vec{x}$ is in W and α is any scalar

Basic Examples

$\boxed{\text{Meta Example:}}$

More examples / Non examples

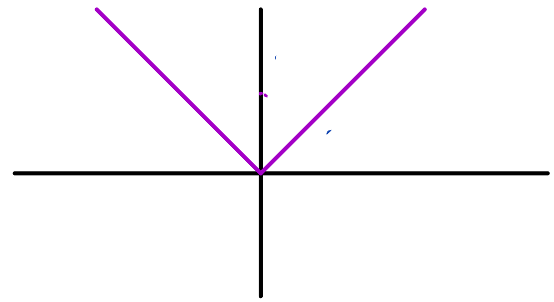
(S1) $\vec{0}$ in W

(S2) Closed under $+$

(S3) ________ scalar mult

1. The line L in $\mathbb{R}^3$ through $(0,0,0)$ with direction $\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$

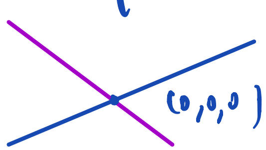
② graph of $|x|$ in $\mathbb{R}$:
$= \{ (x, |x|) : x \text{ in } \mathbb{R} \}$



③ $W = \{ \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_3 = 3 \}$ = plane with equation $z = 3$.

④ $W = \{ \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1, x_2 \text{ are integers} \}$

⑤ Union of 2 different lines in $\mathbb{R}^3$ through $(0,0,0)$



Meta Example 2: The span of a subset

<u>Def</u>: Fix $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in $\mathbb{R}^n$. We write:

$$W = Sp(\vec{v}_1, \ldots, \vec{v}_p) = \text{set of all linear combinations of } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$$

$$= \{ \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_p \vec{v}_p : \alpha_1, \alpha_2, \ldots, \alpha_p \text{ in } \mathbb{R} \}$$

<u>Examples</u>

Theorem 3: The set $W = Sp(\vec{v}_1, \dots, \vec{v}_p)$ in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

Meta example 1: The Null Space of a Matrix

Def: Given A $m\times n$ matrix, we define its Null Space as:

$$\mathcal{N}(A) = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ in } \mathbb{R}^n \;:\; A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \text{ in } \mathbb{R}^m \right\}$$

= Solution set to the homogeneous system with matrix A.

Theorem 3: $\mathcal{N}(A)$ is a subspace of $\mathbb{R}^n$.