Lecture 14: §3.1-2: Intro & Vector Space Properties of $\mathbb{R}^n$
§3.3 Examples of Subspaces of $\mathbb{R}^n$

So far we have seen 2 constructions:

1. (Column) Vectors in $\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^4, \ldots$

2. Solutions to homogeneous systems in $\mathbb{R}^n$ can be written as:

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \alpha_1 \vec{v_1} + \alpha_2 \vec{v_2} + \cdots + \alpha_m \vec{v_m} \qquad (\alpha_1, \ldots \alpha_m \text{ in } \mathbb{R})$$

where $\alpha_1, \alpha_2, \ldots, \alpha_m$ are the m independent variables of $A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

$\operatorname{rank}(A) = n - m$.

Ex $A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ ⤳ x_1, x_3 dep, x_2, x_4 indep

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \overset{\alpha_1}{x_2} \underset{=\vec{v_1}}{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}} + \overset{\alpha_2}{x_4} \underset{=\vec{v_2}}{\begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}}$$

Q: What do these 2 constructions have in common?

A:
- $\vec{0}$ lies in the space
- Add vectors / solutions gives a new vector / solution
- Scalar multiplication preserves vectors / space of solutions

$\Big\}$ basis for vector spaces

Defining properties for $\mathbb{R}^n$

⇝ some will work for abstract vector spaces!

Theorem 1: Write $\mathbb{V} = \mathbb{R}^n$. For $\vec{x}, \vec{y}, \vec{z}$ in $\mathbb{V}$, α, β scalars, we have

① **Closure Properties**:

(C1) $\vec{x}, \vec{y}$ in $\mathbb{V}$, then $\vec{x} + \vec{y}$ in $\mathbb{V}$

(C2) $\vec{x}$ in $\mathbb{V}$, then $\alpha \cdot \vec{x}$ in $\mathbb{V}$

② **Addition Properties**:

(A1) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ (Commutative)

(A2) $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$ (Associative)

(Neutral element) ← (A3) $\vec{0}$ in $\mathbb{V}$ satisfies $\vec{x} + \vec{0} = \vec{0} + \vec{x} = \vec{x}$ for all $\vec{x}$.

(Additive Inverses) ← (A4) given $\vec{x}$ in $\mathbb{V}$ we can find "$-\vec{x}$" in $\mathbb{V}$ with $\vec{x} + (-\vec{x}) = \vec{0}$ (here "$-x$" $= (-1)\vec{x}$)

③ **Scalar Mult. Properties**:

(M1) $\alpha(\beta \vec{x}) = (\alpha\beta)\vec{x}$ (Associative)

(M2) $\alpha(\vec{x} + \vec{y}) = \alpha\vec{x} + \alpha\vec{y}$ (Distributive 1)

(M3) $(\alpha + \beta)\vec{x} = \alpha\vec{x} + \beta\vec{x}$ (—— 2)

(M4) $1\vec{x} = \vec{x}$ for all $\vec{x}$

(A4) follows from (C2)

Subspaces of $\mathbb{R}^n$

Def A subset W of $\mathbb{R}^n$ is a <u>subspace</u> if these 10 properties hold for W

Key Fact: Given a subset W for $\mathbb{R}^n$, we <u>don't need</u> to check <u>all</u> 10 properties to see if W is a subspace.

(A1) $\vec{x}+\vec{y}=\vec{y}+\vec{x}$ True in $\mathbb{R}^n$, so true in W

(A2) $\vec{x}+(\vec{y}+\vec{z})=(\vec{x}+\vec{y})+\vec{z}$ ______

(M1) $\alpha(\beta\vec{x})=(\alpha\beta)\vec{x}$ True in $\mathbb{R}^n$, so true in W

(M2) $\alpha(\vec{x}+\vec{y})=\alpha\vec{x}+\alpha\vec{y}$ ______

(M3) $(\alpha+\beta)\vec{x}=\alpha\vec{x}+\beta\vec{x}$ ______

(M4) $1\vec{x}=\vec{x}$ for all $\vec{x}$ ______

(A4) $\left(\begin{array}{l}\vec{x}+(-\vec{x})=\vec{0}\\ (-\vec{x}) \text{ in } W\end{array}\right)$ follows from (C2) (if x in W, then αx in W)

Conclude: Only need to check (C1), (C2) & (A3).

Theorem 2: A subset $\mathbb{W}$ in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if and only if

(S1) The zero vector lies in $\mathbb{W}$.

(S2) $\vec{x}+\vec{y}$ lies in $\mathbb{W}$ whenever $\vec{x}, \vec{y}$ are in $\mathbb{W}$

(S3) $\alpha\vec{x}$ ____________________ $\vec{x}$ is in $\mathbb{W}$ and α is any scalar

Basic Examples: ① $\mathbb{W} = \{\vec{0}\}$ is a subspace of $\mathbb{R}^n$

② $\mathbb{W} = \mathbb{R}^n$ ____________________

Meta Example: Solution to a homogeneous system in n unknowns

Write $\mathbb{W} := \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : \underset{m\times n}{A} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \text{ in } \mathbb{R}^m \right\}$

(S1) $\underline{x} = \vec{0}$ is a solution

(S2) If $\underline{x}$, $\underline{y}$ are solutions then $A(\underline{x}+\underline{y}) = A\underline{x} + A\underline{y} = \vec{0} + \vec{0} = \vec{0}$

so $\underline{x}+\underline{y}$ is also a solution

(S3) If $\underline{x}$ is a solution $A(\alpha\underline{x}) = \alpha(A\underline{x}) = \alpha\vec{0} = \vec{0}$

so $\alpha\underline{x}$ is also a solution.

More examples / Non examples

(S1) $\vec{0}$ in W
(S2) Closed under $+$
(S3) ______ scalar mult

① The line L in $\mathbb{R}^3$ through $(0,0,0)$ with direction $\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$

(S1) $(0,0,0)$ in L $\quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$

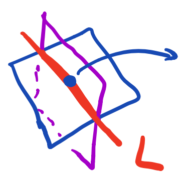
(S2) $P = (p_1, p_2, p_3)$, $Q = (q_1, q_2, q_3)$ $\rightsquigarrow$ $\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = t_1 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = t_2 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$, then:

$$\begin{bmatrix} p_1+q_1 \\ p_2+q_2 \\ p_3+q_3 \end{bmatrix} = \underbrace{(t_1+t_2)}_{\text{new scalar}} \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$ so $\vec{OP} + \vec{OQ}$ is in L.

(S3) $P = (p_1, p_2, p_3)$, α scalar

$$\alpha \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \underbrace{\alpha\, t_1}_{\text{new scalar}} \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$ so $\alpha \vec{OP}$ is in L.

Soln 2: $\begin{cases} x + y + z = 0 \\ x + 2y - z = 0 \end{cases}$

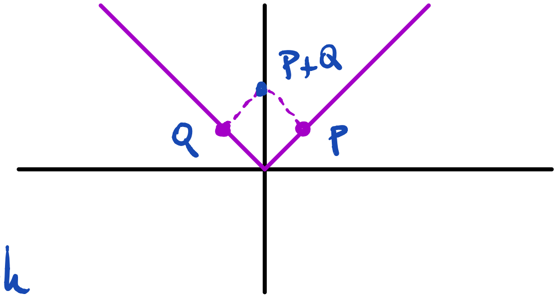


$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

L is the solution set to a homogeneous system $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so the meta example applies.

- Same ideas work for every line in $\mathbb{R}^3$ through $(0,0,0)$

② graph of $|x|$ in $\mathbb{R}$: $= \{ (x, |x|) : x \text{ in } \mathbb{R} \}$



$\rightsquigarrow$ NOT a subspace of $\mathbb{R}^2$

(S1) ok $(0,0)$ in the graph

(S2) fails $P = (1,1)$, $Q = (1,-1)$ in the graph BUT $P+Q = (0,2)$ is not

③ $W = \left\{ \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_3 = 3 \right\}$ = plane with equation $z = 3$.

(S1) fails, so W is not a subspace of $\mathbb{R}^3$

Same is true for any plane in $\mathbb{R}^3$ not containing $(0,0,0)$. $\rightsquigarrow$ NOT a subspace of $\mathbb{R}^2$.

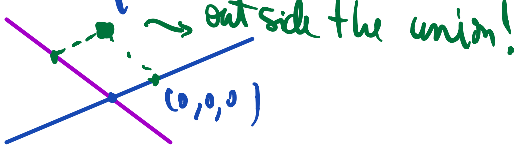
④ $W = \left\{ \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1, x_2 \text{ are integers} \right\}$ $\rightsquigarrow$ NOT a subspace of $\mathbb{R}^2$.

(S1) holds

(S2) —— (sum of integers is always an integer)

(S3) does not hold $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in W $\alpha = \frac{1}{2}$ but $\frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ is not in W.

⑤ Union of 2 different lines in $\mathbb{R}^3$ through $(0,0,0)$ $\rightsquigarrow$ NOT a subspace of $\mathbb{R}^3$



(S1) & (S3) are true
(S2) is not.

Meta Example 2: The span of a subset

Def: Fix $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in $\mathbb{R}^n$. We write:

$$W = Sp(\vec{v}_1, \ldots, \vec{v}_p) = \text{set of all linear combinations of } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$$

$$= \{ \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_p \vec{v}_p : \alpha_1, \alpha_2, \ldots, \alpha_p \text{ in } \mathbb{R} \}$$

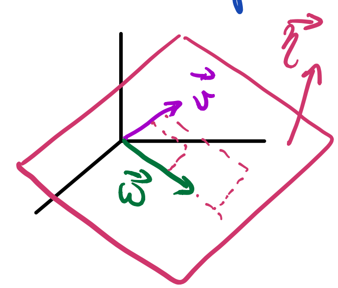
Examples 0) $Sp(\vec{0}) = \{\vec{0}\}$

1) $\vec{v}$ in $\mathbb{R}^3$, $\vec{v} \neq \vec{0}$ $\leadsto$ $W = Sp(\vec{v}) = \{\alpha \vec{v}\}$ = line in $\mathbb{R}^3$ with direction $\vec{v}$ through $(0,0,0)$

3) $\vec{v}, \vec{w}$ in $\mathbb{R}^3$, $\vec{v}, \vec{w}$ not proportional, $\vec{v}, \vec{w} \neq \vec{0}$

$$Sp(\vec{v}, \vec{w}) = \{\alpha \vec{v} + \beta \vec{w}\}$$

= plane throught $(0,0,0)$ with normal $\vec{n} = \vec{v} \times \vec{w}$.



Eg: $\vec{v} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad \vec{n} = \vec{v} \times \vec{w} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$ $\leadsto$ $\boxed{-x+2y-z=0}$

Q Other Soln? $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2\alpha + \beta \\ -\alpha + \beta \\ \beta \end{bmatrix}$ Given (x, y, z) we want to solve for α, β

$$\left[\begin{array}{cc|c} -2 & 1 & x \\ -1 & 1 & y \\ 0 & 1 & z \end{array}\right] \xrightarrow[R_1 \to -R_1]{R_2 \leftrightarrow R_1} \left[\begin{array}{cc|c} 1 & -1 & -y \\ -2 & 1 & x \\ 0 & 1 & z \end{array}\right] \xrightarrow{R_2 \to R_2 + 2R_1} \left[\begin{array}{cc|c} 1 & -1 & -y \\ 0 & -1 & x-2y \\ 0 & 1 & z \end{array}\right] \xrightarrow[R_2 \to -R_2]{R_3 \to R_3 + R_2} \left[\begin{array}{cc|c} 1 & -1 & -y \\ 0 & 1 & -x+2y \\ 0 & 0 & x-2y+z \end{array}\right]$$

We can solve for (a,b) if and only if $\boxed{x - 2y + z = 0}$

Theorem 3: The set $W = Sp(\vec{v}_1, ---, \vec{v}_p)$ in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

Why? Let's check the 3 properties for subspaces.

(S1) $\vec{0} = 0 \cdot \vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_p$ in W ✓

(S2) $\vec{u}$ in W $\quad \vec{u} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_p \vec{v}_p$ for some scalars $\alpha_1, \ldots, \alpha_p$

$\vec{w}$ in W $\quad + \quad \vec{w} = \beta_1 \vec{v}_1 + \beta_2 \vec{v}_2 + \cdots + \beta_p \vec{v}_p$ $\quad \beta_1, \ldots, \beta_p$

$$\vec{u} + \vec{w} = \underbrace{(\alpha_1+\beta_1)}_{\text{scalar}}\vec{v}_1 + \underbrace{(\alpha_2+\beta_2)}_{\text{scalar}}\vec{v}_2 + \cdots + \underbrace{(\alpha_p+\beta_p)}_{\text{scalar}}\vec{v}_p$$

Add Term-by-Term

So $\vec{u}+\vec{w}$ is in $Sp(\vec{v}_1, \ldots, \vec{v}_p)$

(S3) $$\alpha \vec{w} = \underbrace{(\alpha\beta_1)}_{\text{scalar}} \vec{v}_1 + \underbrace{(\alpha\beta_2)}_{\text{scalar}} \vec{v}_2 + \cdots + \underbrace{(\alpha\beta_p)}_{\text{scalar}} \vec{v}_p$$

So $\alpha \vec{w}$ is in $Sp(\vec{v}_1, \ldots, \vec{v}_p)$ if $\vec{w}$ is.

Example from early shows how each $Sp(\vec{v}_1, \ldots, \vec{v}_p)$ is again a solution set to a homogeneous system. (The same method will work to find its equations).

Meta example 1: The Null Space of a Matrix

Def: Given A $m\times n$ matrix, we define its Null Space as:

$$\mathcal{N}(A) = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ in } \mathbb{R}^n : A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \text{ in } \mathbb{R}^m \right\}$$

= Solution set to the homogeneous system with matrix A.

Theorem 3: $\mathcal{N}(A)$ is a subspace of $\mathbb{R}^n$.

Example: $A = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 3 & 2 & 8 & 5 \\ -1 & -2 & -4 & 1 \end{bmatrix}$ $\rightsquigarrow$ $[A \mid 0] \overset{GJ}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ REF $\begin{cases} x_3, x_4 \text{ indep} \\ x_1, x_2 \text{ dep} \end{cases}$

$\rightsquigarrow$ $x_1 = -2x_3 - 3x_4$
$x_2 = -x_3 + 2x_4$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 - 3x_4 \\ -x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

x_3, x_4 any numbers!

Conclude $\mathcal{N}(A) = \mathrm{Sp}\left(\begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right)$

Note: These 2 vectors are l.i.! (look at the last 2 entries)

In general: $\mathcal{N}(A)$ will be the span of a finite number of vectors, as many as the number of independent variables.