Lecture 15: §3.3 More examples of subspaces of $\mathbb{R}^n$
§3.4 Bases for Subspaces

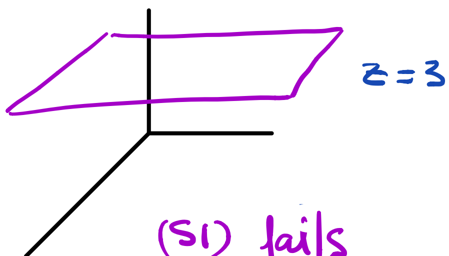
Recall: A subset W of $\mathbb{R}^n$ is a (vector) subspace if it satisfies:

(S1) $\vec{0}$ in W

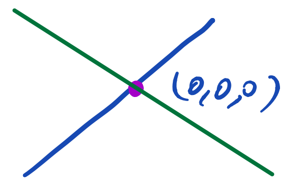
(S2) If $\vec{v}$, $\vec{w}$ in W, then $\vec{v}+\vec{w}$ is also in W

(S3) If $\vec{v}$ in W & α in $\mathbb{R}$, then $\alpha\vec{v}$ is also in W.

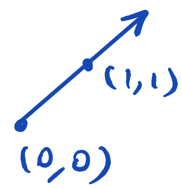
- Non examples



(S1) fails



(S2) fails



(S3) fails

- $\boxed{\text{2 Main examples}}$

① $\mathcal{N}(A) = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ in } \mathbb{R}^n : A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{0} \text{ in } \mathbb{R}^m \right\}$ Null Space of A

(A $m\times n$ fixed)

[Solutions to homogeneous systems]

② $Sp(\vec{v}_1, \dots, \vec{v}_p)$ = linear combinations of vectors $\vec{v}_1, \dots, \vec{v}_p$

$= \{\alpha_1\vec{v}_1 + \dots + \alpha_p\vec{v}_p : \alpha_1, \dots, \alpha_p \text{ in } \mathbb{R}\}$

($\vec{v}_1, \dots, \vec{v}_p$ fixed vectors)

[Spans of vectors]

The Range of an $m \times n$ matrix A

Column Space

Def: The range of A is $\boxed{R(A) = \left\{ \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \text{ in } \mathbb{R}^m : \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ for some } \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ in } \mathbb{R}^n \right\}}$

Recall $A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \operatorname{col}_1(A) + \cdots + x_n \operatorname{col}_n(A)$.

Theorem: $R(A) = Sp(n \text{ columns of } A)$ is a subspace of $\mathbb{R}^m$

Example: $\underset{(3 \times 4)}{A} = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -1 & 4 & 0 \\ 1 & 1 & 5 & -3 \end{bmatrix}$

Example 2:

$$\underset{(3\times 4)}{A} = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -1 & 4 & 0 \\ 1 & 1 & 5 & -1 \end{bmatrix}$$

The Row Space of an $m \times n$ matrix A

Def: The <u>Row Space of A</u> is $\mathrm{Row}(A) = \mathrm{Sp}(m \text{ rows of } A)$ in $\mathbb{R}^n$.

Obs: $\mathrm{Row}(A) = \mathrm{Sp}(\text{Columns of } A^t) = \mathcal{R}(A^t)$

Q: What happens under elementary row operations?

A: Same row space!

Advantage: We can use <u>row operations</u> to find a better set of generators for $\mathrm{Row}(A)$ & in general, for $\mathrm{Sp}(\vec{v}_1, \dots, \vec{v}_m)$ in $\mathbb{R}^n$.

Example $\mathbb{V} = \mathrm{Sp}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -4 \end{bmatrix} \right)$ (4 generators)

Theorem 2: If A & B are row equivalent, then $\text{Row}(A) = \text{Row}(B)$

Spanning Sets & Bases

Def Fix a subspace $\mathbb{V}$ of $\mathbb{R}^n$ & a subset $S = \{\vec{v}_1, \dots, \vec{v}_p\}$ of $\mathbb{R}^n$.

1. We say S is a spanning set for $\mathbb{V}$ (or S spans $\mathbb{V}$) if

2. S is a minimal spanning set for $\mathbb{V}$ if:

Def: A basis for $\mathbb{V}$ is a finite minimal spanning set for $\mathbb{V}$

Next Time: Basis B = (1) B spans $\mathbb{V}$
& (2) B is linearly independent

$$\boxed{\text{Examples}}$$

1. $\mathbb{V} = \mathbb{R}^3 = \mathrm{Sp}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$