Lecture 15: §3.3 More examples of subspaces of $\mathbb{R}^n$
§3.4 Bases for Subspaces

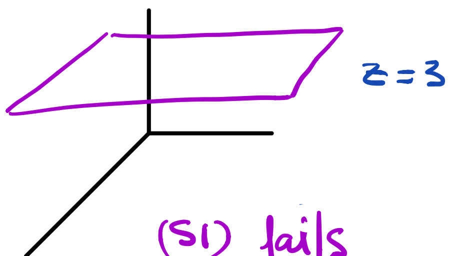
Recall: A subset W of $\mathbb{R}^n$ is a (vector) subspace if it satisfies:

(S1) $\vec{0}$ in W

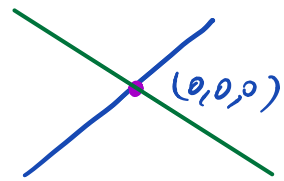
(S2) If $\vec{v}, \vec{w}$ in W, then $\vec{v}+\vec{w}$ is also in W

(S3) If $\vec{v}$ in W & α in $\mathbb{R}$, then $\alpha\vec{v}$ is also in W.

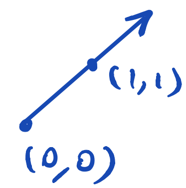
- Non examples



(S1) fails



(S2) fails



(S3) fails

- $\boxed{\text{2 Main examples}}$

① $\mathcal{N}(A) = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ in } \mathbb{R}^n : A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{0} \text{ in } \mathbb{R}^m \right\}$ NullSpace of A

(A $m\times n$ fixed)

[Solutions to homogeneous systems]

② $Sp(\vec{v}_1, \ldots, \vec{v}_p)$ = linear combinations of vectors $\vec{v}_1, \ldots, \vec{v}_p$

$= \{ \alpha_1\vec{v}_1 + \cdots + \alpha_p\vec{v}_p : \alpha_1, \ldots, \alpha_p \text{ in } \mathbb{R} \}$

($\vec{v}_1, \ldots, \vec{v}_p$ fixed vectors)

[Spans of vectors]

The Range of an $m \times n$ matrix A

Column Space

Def: The range of A is
$$\boxed{\mathcal{R}(A) = \left\{ \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \text{ in } \mathbb{R}^m : \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ for some } \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ in } \mathbb{R}^n \right\}}$$

Recall
$$A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \operatorname{col}_1(A) + \cdots + x_n \operatorname{col}_n(A).$$

Theorem: $\mathcal{R}(A) = \operatorname{Sp}(n \text{ columns of } A)$ is a subspace of $\mathbb{R}^m$

Example:
$$\underset{(3\times 4)}{A} = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -1 & 4 & 0 \\ 1 & 1 & 5 & -3 \end{bmatrix} \rightsquigarrow \mathcal{R}(A) = \operatorname{Sp}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \right) \text{ in } \mathbb{R}^3$$

Q: What is this subspace? Given $\underline{y}$ in $\mathcal{R}(A)$, write $A\underline{x} = \underline{y}$ & solve for $\underline{x}$.

$$[A \mid \underline{y}] = \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & y_1 \\ 2 & -1 & 4 & 0 & y_2 \\ 1 & 1 & 5 & -3 & y_3 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & y_1 \\ 0 & 1 & 2 & -2 & y_2 - 2y_1 \\ 0 & 2 & 4 & -4 & y_3 - y_1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_2]{} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & y_1 \\ 0 & 1 & 2 & -2 & y_2 - 2y_1 \\ 0 & 0 & 0 & 0 & 3y_1 - 2y_2 + y_3 \end{array}\right]$$

EF

Compatible if and only if $3y_1 - 2y_2 + y_3 = 0$ plane in $\mathbb{R}^3$ $\rightsquigarrow y_3 = -3y_1 + 2y_2$

$$\rightsquigarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ -3y_1 + 2y_2 \end{bmatrix} = y_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + y_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \rightsquigarrow \boxed{\mathcal{R}(A) = \operatorname{Sp}\left(\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right) = \mathcal{N}([3 \; -2 \; 1])}$$

Example 2: $\underset{(3\times 4)}{A} = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -1 & 4 & 0 \\ 1 & 1 & 5 & -1 \end{bmatrix} \rightsquigarrow R(A) = \mathrm{Sp}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)$

$$[A \mid \underline{y}] = \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & y_1 \\ 2 & -1 & 4 & 0 & y_2 \\ 1 & 1 & 5 & -1 & y_3 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & y_1 \\ 0 & 1 & 2 & -2 & y_2 - 2y_1 \\ 0 & 2 & 4 & -2 & y_3 - y_1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_2]{} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & y_1 \\ 0 & 1 & 2 & -2 & y_2 - 2y_1 \\ 0 & 0 & 0 & 2 & 3y_1 - 2y_2 + y_3 \end{array}\right]$$

$$\xrightarrow[R_3 \to \frac{1}{2} R_3]{} \underbrace{\left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & y_1 \\ 0 & 1 & 2 & -2 & y_2 - 2y_1 \\ 0 & 0 & 0 & 1 & 3y_1 - 2y_2 + y_3 \end{array}\right]}_{EF}$$

Always consistent, for all $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Conclude: $R(A) = \mathbb{R}^3 = \mathrm{Sp}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$

In general: Range are either $\mathbb{R}^n$ or it is determined by a linear system of equations.

Method $\left[\begin{array}{c|c} A & \begin{array}{c} y_1 \\ \vdots \\ y_n \end{array} \end{array}\right] \sim \left[\begin{array}{c|c} A' & \text{linear eqns} \\ \hline \begin{array}{ccc} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{array} & \boxed{\text{linear eqns}} \end{array}\right]$ where

A' is REF with no zero rows.

equations for $R(A)$

The Row Space of an $m \times n$ matrix A

Def: The Row Space of A is $\text{Rows}(A) = \text{Sp}(m \text{ rows of } A)$ in $\mathbb{R}^n$.

Obs: $\text{Rows}(A) = \text{Sp}(\text{Columns of } A^t) = \mathcal{R}(A^t)$

Example: $A = \begin{bmatrix} 1 & 2 & 4 \\ 5 & 7 & 8 \end{bmatrix}$

$\mathcal{R}(A) = \text{Sp}\left(\begin{bmatrix} 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix} \right)$ in $\mathbb{R}^2$

$\text{Rows}(A) = \text{Sp}\left(\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix} \right)$ in $\mathbb{R}^3$

Q: What happens under elementary row operations?

A: Same row space!

Advantage: We can use row operations to find a better set of generators for $\text{Row}(A)$ & in general, for $\text{Sp}(\vec{v}_1, \dots, \vec{v}_m)$ in $\mathbb{R}^n$.

$$A = \begin{bmatrix} v_1^t \\ \vdots \\ v_m^t \end{bmatrix} \overset{G\text{-}J}{\sim} B = \begin{bmatrix} w_1^t \\ \vdots \\ w_r^t \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix} \rightsquigarrow \boxed{W = \text{Rows}(A) = \text{Sp}(\vec{w}_1, \dots, \vec{w}_r)}$$

Example $W = \mathrm{Sp}\left(\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\3\\5\end{bmatrix}, \begin{bmatrix}3\\5\\6\end{bmatrix}, \begin{bmatrix}-1\\-1\\-4\end{bmatrix}\right)$ (4 generators)

Step 1: Write vectors as the ROWS of a matrix A

$$\underset{4\times 3}{A} = \begin{bmatrix}1&2&1\\2&3&5\\3&5&6\\-1&-1&-4\end{bmatrix}$$

Step 2: Find $A \underset{\text{row}}{\sim} B$ with B in REF

$$A = \begin{bmatrix}1&2&1\\2&3&5\\3&5&6\\-1&-1&-4\end{bmatrix} \xrightarrow[\substack{R_2\to R_2-2R_1\\ R_3\to R_3-3R_2\\ R_4\to R_1+R_1}]{} \begin{bmatrix}1&2&1\\0&-1&3\\0&-1&3\\0&1&-3\end{bmatrix} \xrightarrow[\substack{R_3\to R_3-R_2\\ R_4\to R_4+R_2}]{} \begin{bmatrix}1&2&1\\0&-1&3\\0&0&0\\0&0&0\end{bmatrix} \xrightarrow[\substack{R_2\to -R_2\\ R_1\to R_1-2R_2}]{} \begin{bmatrix}1&0&7\\0&1&-3\\0&0&0\\0&0&0\end{bmatrix} = B$$

REF

Step 3: Pick non-zero rows of B, & transpose them.

$$W = \mathrm{Sp}\left(\underbrace{\begin{bmatrix}1\\0\\7\end{bmatrix}}_{\text{"}\vec{w}_1\text{"}} \underbrace{\begin{bmatrix}0\\1\\-3\end{bmatrix}}_{\text{"}\vec{w}_2\text{"}}\right)$$

2 generators instead of 4

Q: Can we do better?

A: NO! 2 is the minimal number we need ($\vec{w}_1$ & $\vec{w}_2$ are l.i)

Advantage

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = a\begin{bmatrix}1\\0\\7\end{bmatrix} + b\begin{bmatrix}0\\1\\-3\end{bmatrix} = \begin{bmatrix}a\\b\\7a-3b\end{bmatrix} \rightsquigarrow \begin{cases}a = x\\ b = y\\ 7a-3b = z\end{cases}$$

$\rightsquigarrow$ Eqn: $\boxed{7x-3y-z=0}$ $\rightsquigarrow W = N([7\ -3\ -1])$

Theorem 2: If A & B are row equivalent, then $\text{Rows}(A) = \text{Rows}(B)$

Why? It is enough to check each elementary row operation preserves row spaces.

$$A = A_1 \xrightarrow{\text{Elem}} A_2 \xrightarrow{\text{Elem}} A_3 \xrightarrow{\text{Elem}} \cdots \xrightarrow{\text{Elem}} A_k = B$$

Then: $\text{Rows}(A) = \text{Rows}(A_2) = \cdots = \text{Rows}(A_k) = \text{Rows}(B)$

(E1) Swapping 2 rows clearly preserves the row space.

(E2) Multiplying a row (e.g. R_1) by a nonzero scalar $\alpha \neq 0$:

$$\underline{x} \text{ in } Sp(R_1^t, R_2^t, \ldots, R_n^t) \stackrel{?}{=} Sp(\alpha R_1^t, R_2^t, \ldots, R_n^t)$$

$$\underline{x} = \beta_1 R_1^t + \beta_2 R_2^t + \cdots + \beta_n R_n^t = \frac{\beta_1}{\alpha}(\alpha R_1^t) + \beta_2 R_2^t + \cdots + \beta_n R_n^t$$

OK because $\alpha \neq 0$.

(E3) Adding to a row (say, R_1) a scalar multiply of a different row (R_2)

$$Sp(R_1^t, R_2^t, \ldots R_n^t) \stackrel{?}{=} Sp(R_1^t + \alpha R_2^t, R_2^t, \ldots, R_n^t)$$

$$\beta_1 R_1^t + \beta_2 R_2^t + \cdots + \beta_n R_n^t = \beta_1(R_1^t + \alpha R_2^t) + (\beta_2 - \alpha\beta_1) R_2^t + \cdots + \beta_n R_n^t$$

Spanning Sets & Bases

<u>Def</u> Fix a subspace $\mathbb{V}$ of $\mathbb{R}^n$ & a subset $S = \{\vec{v}_1, \dots, \vec{v}_p\}$ of $\mathbb{R}^n$.

1. We say S is a <u>spanning set</u> for $\mathbb{V}$ (or S spans $\mathbb{V}$) if

$$\mathbb{V} = Sp(\vec{v}_1, \dots, \vec{v}_p)$$

2. S is a <u>minimal spanning set</u> for $\mathbb{V}$ if:

   (i) S spans $\mathbb{V}$

   (ii) Any proper subset of S does not span $\mathbb{V}$.

for all $i = 1, \dots, p$: $Sp(\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_p\}) \neq \mathbb{V}$

we removed $\vec{v}_i$.

<u>Def</u>: A <u>basis</u> for $\mathbb{V}$ is a finite minimal spanning set for $\mathbb{V}$

<u>Next Time</u>: Basis B = (1) B spans $\mathbb{V}$

& (2) B is linearly independent

Examples

(1) $\mathbb{V} = \mathbb{R}^3 = Sp\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$ $\qquad \begin{bmatrix}x\\y\\z\end{bmatrix} = x\begin{bmatrix}1\\0\\0\end{bmatrix} + y\begin{bmatrix}0\\1\\0\end{bmatrix} + z\begin{bmatrix}0\\0\\1\end{bmatrix}$

- $\{e_1, e_2, e_3\}$ spans $\mathbb{R}^3$
- ________ minimal spanning set
- $Sp\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right) = \left\{ \begin{bmatrix}a\\b\\0\end{bmatrix} : a, b \text{ in } \mathbb{R} \right\} =$ (plane with equation $z=0$) $\neq \mathbb{R}^3$

  XY-plane
- $Sp\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right) =$ (plane with equation $y=0$) $\neq \mathbb{R}^3$

  XZ-plane
- $Sp\left(\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right) =$ (plane with equation $x=0$) $\neq \mathbb{R}^3$

  YZ-plane

<u>Conclude</u> $\{e_1, e_2, e_3\}$ is a <u>basis</u> for $\mathbb{R}^3$ (<u>Name</u>: standard basis)

<u>Another basis</u>: $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}$ $\qquad \begin{bmatrix}x\\y\\z\end{bmatrix} = (x-y)\begin{bmatrix}1\\0\\0\end{bmatrix} + (y-z)\begin{bmatrix}1\\1\\0\end{bmatrix} + z\begin{bmatrix}1\\1\\1\end{bmatrix}$

- Any 2 vectors are coplanar, so they cannot generate $\mathbb{R}^3$.