Lecture 16: §3.4 Bases for Subspaces

Last Time: 4 main examples of subspaces $\mathbb{V}$ of $\mathbb{R}^N$

$$\begin{cases} (S1) & \vec{0} \text{ in } \mathbb{V} \\ (S2) & \text{closed under } + \\ (S3) & \text{—— scalar mult.} \end{cases}$$

① $\mathcal{N}(A) = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ in } \mathbb{R}^n : A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{0} \text{ in } \mathbb{R}^m \right\}$ NullSpace of A

(A $m \times n$ fixed)

[Solutions to homogeneous systems]

② $Sp(\vec{v}_1, \ldots, \vec{v}_p)$ = linear combinations of vectors $\vec{v}_1, \ldots, \vec{v}_p$

($\vec{v}_1, \ldots, \vec{v}_p$ fixed vectors in $\mathbb{R}^n$) $= \{\alpha_1 \vec{v}_1 + \cdots + \alpha_p \vec{v}_p : \alpha_1, \ldots, \alpha_p \text{ in } \mathbb{R}\}$

[Spans of vectors]

③ A $m \times n$ fixed $\quad \mathcal{R}(A) = Sp(\mathrm{col}_1(A), \ldots, \mathrm{col}_n(A))$ (in $\mathbb{R}^m$)

[Range or Column Space]

④ A $m \times n$ fixed $\quad \mathrm{Rows}(A) = Sp(\mathrm{Row}_1(A)^t, \ldots, \mathrm{Row}_m(A)^t)$ (in $\mathbb{R}^n$)

[Row Space]

- **Spanning Set for $\mathbb{V}$**: a list $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ in $\mathbb{R}^N$ with $\mathbb{V} = Sp(\vec{v}_1, \ldots, \vec{v}_p)$
- **Bases for $\mathbb{V}$**: A minimal spanning set S

↳ If any $\vec{v}_i$ is remove from S, we no longer span.

EXAMPLES

① Check if $\left\{ \overset{\vec{v}_1}{\begin{bmatrix}1\\2\\3\end{bmatrix}} \overset{\vec{v}_2}{\begin{bmatrix}-1\\0\\-7\end{bmatrix}} \overset{\vec{v}_3}{\begin{bmatrix}2\\7\\0\end{bmatrix}} \right\}$ spans $\mathbb{R}^3$.

② Check if $S = \left\{ \begin{bmatrix}1\\2\\3\end{bmatrix} \begin{bmatrix}-1\\0\\-7\end{bmatrix} \begin{bmatrix}2\\8\\1\end{bmatrix} \right\}$ spans $\mathbb{R}^3$.

③ $$A = \begin{array}{c} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -1 & 4 & 0 \\ 1 & 1 & 5 & -3 \end{bmatrix} \\ \begin{matrix} \Vert & \Vert & \Vert & \Vert \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{matrix} \end{array} \rightsquigarrow \boxed{R(A) = Sp\left(\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right) = N\left(\begin{bmatrix} 3 & -2 & 1 \end{bmatrix} \right)}$$

$$\boxed{\text{Best spanning sets = Minimal ones (Bases!)}}$$

<u>Example</u>: $\mathbb{W} = Sp\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}-1\\0\\-7\end{bmatrix}, \begin{bmatrix}2\\7\\0\end{bmatrix}\right) = Sp\left(\begin{bmatrix}1\\0\\7\end{bmatrix}, \begin{bmatrix}0\\1\\-2\end{bmatrix}\right) = Sp\left(\begin{bmatrix}-1\\0\\-7\end{bmatrix}, \begin{bmatrix}2\\7\\0\end{bmatrix}\right)$

<u>Why?</u>

<u>Q</u>: How to get a minimal spanning set for $\mathbb{W} = Sp(\vec{v}_1, \dots, \vec{v}_p)$?

<u>A</u>: Use linear dependencies!

$\boxed{\text{ALGORITHM: From Spanning Set to Basis}}$

INPUT: $S = \{\vec{v}_1, \dots, \vec{v}_p\}$ a spanning set of W (subspace of $\mathbb{R}^n$)

OUTPUT: S' = subset of S that is a basis for W (= minimal spanning set)

Step 1 Is S l.i ?
- If YES, then $\boxed{\text{output } S}$
- If NO, find a nontrivial solution $(a_1, \dots a_p)$ to $a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_p\vec{v}_p = \vec{0}$

Pick smallest index i with $a_i \neq 0$

New $S = \{\vec{v}_1, \dots \vec{v}_{i-1}, \vec{v}_{i+1}, \dots \vec{v}_p\}$

$= S \setminus \{v_i\}$

Step 2 Repeat Step 1 for New S, ... at some point, we are l.i & we exit.

The algorithm gives a new characterization for bases

Proposition: A set $B = \{\vec{v}_1, \dots \vec{v}_p\}$ is a basis for $\mathbb{V}$ if

(B1) B spans $\mathbb{V}$ & (B2) B is l.i.

⚠ $\{\vec{0}\}$ is l.d. so $\mathbb{V} = \{\vec{0}\}$ has no basis! It is the only subspace of $\mathbb{V}$ without a basis.

A Basis $\mathcal{B}$ gives a new coordinate system for $\mathbb{V}$

Theorem Uniqueness of representation.

Pick a subspace $\mathbb{V} \neq \{\vec{0}\}$ of $\mathbb{R}^n$ with basis $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_p\}$.
Then, any $\vec{v}$ in $\mathbb{V}$ can be represented in a <u>unique way</u> as a linear combination of $\vec{v}_1, \ldots, \vec{v}_p$. That is, we can find unique scalars $a_1, \ldots, a_p$ so that:

$$\vec{v} = a_1 \vec{v}_1 + \cdots + a_p \vec{v}_p$$

Name: $\begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix} = [v]_{\mathcal{B}} =$ coordinates for $\vec{v}$ with respect to the basis $\mathcal{B}$.

Theorem Uniqueness of representation.

Pick a subspace $W \neq \{\vec{0}\}$ of $\mathbb{R}^n$ with basis $B = \{\vec{v}_1, \dots, \vec{v}_p\}$.
Then, any $\vec{v}$ in W can be represented in a unique way as a linear combination of $\vec{v}_1, \dots, \vec{v}_p$. That is, we can find unique scalars $a_1, \dots, a_p$ so that:

$$\boxed{\vec{v} = a_1 \vec{v}_1 + \dots + a_p \vec{v}_p} \qquad (*)$$

given

- Why is this true?