Lecture 16 : §3.4 Bases for Subspaces

Last Time: 4 main examples of subspaces $\mathbb{V}$ of $\mathbb{R}^N$

$$\begin{cases} (S1) & \vec{0} \text{ in } \mathbb{V} \\ (S2) & \text{closed under } + \\ (S3) & \text{——— scalar mult.} \end{cases}$$

① $\mathcal{N}(A) = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ in } \mathbb{R}^n : A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{0} \text{ in } \mathbb{R}^m \right\}$ NullSpace of A

(A $m \times n$ fixed)

[Solutions to homogeneous systems]

② $Sp(\vec{v}_1, \ldots, \vec{v}_p)$ = linear combinations of vectors $\vec{v}_1, \ldots, \vec{v}_p$

($\vec{v}_1, \ldots, \vec{v}_p$ fixed vectors in $\mathbb{R}^n$) $= \{ \alpha_1 \vec{v}_1 + \cdots + \alpha_p \vec{v}_p : \alpha_1, \ldots, \alpha_p \text{ in } \mathbb{R} \}$

[Spans of vectors]

③ A $m \times n$ fixed $\quad \mathcal{R}(A) = Sp(\mathrm{col}_1(A), \ldots, \mathrm{col}_n(A))$ (in $\mathbb{R}^m$) [Range or Column Space]

④ A $m \times n$ fixed $\quad \mathrm{Row}(A) = Sp(\mathrm{Row}_1(A)^t, \ldots, \mathrm{Row}_m(A)^t)$ (in $\mathbb{R}^n$) [Row Space]

- **Spanning Set for $\mathbb{V}$**: a list $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ in $\mathbb{R}^N$ with $\mathbb{V} = Sp(\vec{v}_1, \ldots, \vec{v}_p)$
- **Bases for $\mathbb{V}$**: A minimal spanning set S

↳ If any $\vec{v}_i$ is remove from S, we no longer span.

EXAMPLES

① Check if $\left\{ \underbrace{\begin{bmatrix}1\\2\\3\end{bmatrix}}_{\vec{v}_1} \underbrace{\begin{bmatrix}-1\\0\\-7\end{bmatrix}}_{\vec{v}_2} \underbrace{\begin{bmatrix}2\\7\\0\end{bmatrix}}_{\vec{v}_3} \right\}$ spans $\mathbb{R}^3$.

<u>Soln</u>: Want to check if any $\vec{v} = \begin{bmatrix}x\\y\\z\end{bmatrix}$ can be written as $\vec{v} = \overbrace{a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3}^{\text{unknowns}}$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & x \\ 2 & 0 & 7 & y \\ 3 & -7 & 0 & z \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 2 & x \\ 0 & 2 & 3 & y-2x \\ 0 & -4 & -6 & z-3x \end{array}\right] \xrightarrow[R_2 \to \frac{1}{2}R_2]{R_3 \to R_3 + 2R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & x \\ 0 & 1 & 3/2 & \frac{y}{2}-x \\ 0 & 0 & 0 & z+2y-7x \end{array}\right]$$

The system is compatible if & only if we have no row $[0\ 0\ 0 \mid *]$ with $* \neq 0$

So $\boxed{z+2y-7x = 0.}$

<u>Conclusion</u>: $\mathbb{V} = Sp(\vec{v}_1, \vec{v}_2, \vec{v}_3) \neq \mathbb{R}^3$. In fact, $\mathbb{V}$ = plane with eqn $z+2y-7x=0$

② Check if $S = \left\{ \begin{bmatrix}1\\2\\3\end{bmatrix} \begin{bmatrix}-1\\0\\-7\end{bmatrix} \begin{bmatrix}2\\8\\1\end{bmatrix} \right\}$ spans $\mathbb{R}^3$.

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & x \\ 2 & 0 & 8 & y \\ 3 & -7 & 1 & z \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 2 & x \\ 0 & 2 & 4 & y-2x \\ 0 & -4 & -5 & z-3x \end{array}\right] \xrightarrow{R_3 \to R_3 + 2R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & x \\ 0 & 2 & 4 & y-2x \\ 0 & 0 & 3 & z+2y-7x \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3/3]{R_2 \to R_2/2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & x \\ 0 & 1 & 2 & y/2 - x \\ 0 & 0 & 1 & z/3 + 2/3\, y - 7/3\, x \end{array}\right] \text{EF}$$

Always compatible!

So S spans $\mathbb{R}^3$ (it's a basis!)

③ $A = \begin{bmatrix}1 & -1 & 1 & 1\\ 2 & -1 & 4 & 0\\ 1 & 1 & 5 & -3\end{bmatrix}$ $\rightsquigarrow$ $\boxed{R(A) = Sp\left(\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}\right) = N([3\ -2\ 1])}$

$\quad\quad = \vec{v}_1 \quad = \vec{v}_2 \quad = \vec{v}_3 \quad = \vec{v}_4$

also a basis

- $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ spans $R(A)$ (column space!)
- Not minimal: $\vec{v}_1$ in $Sp(\vec{v}_2, \vec{v}_3, \vec{v}_4)$ $\rightsquigarrow$ don't need $\vec{v}_1$
- $\begin{bmatrix}1\\2\\1\end{bmatrix} \overset{?}{=} a\begin{bmatrix}-1\\-1\\1\end{bmatrix} + b\begin{bmatrix}1\\4\\5\end{bmatrix} + c\begin{bmatrix}1\\0\\-3\end{bmatrix}$

$$\left[\begin{array}{ccc|c}-1 & 1 & 1 & 1\\ -1 & 4 & 0 & 2\\ 1 & 5 & -3 & 1\end{array}\right] \xrightarrow[R_1 \leftrightarrow R_3]{} \left[\begin{array}{ccc|c}1 & 5 & -3 & 1\\ -1 & 4 & 0 & 2\\ -1 & 1 & 1 & 1\end{array}\right] \xrightarrow[\substack{R_2 \to R_2 + R_1\\ R_3 \to R_3 + R_1}]{} \left[\begin{array}{cccc}1 & 5 & -3 & 1\\ 0 & 9 & -3 & 3\\ 0 & 6 & -2 & 2\end{array}\right] \xrightarrow[\substack{R_2 \to R_2/3\\ R_3 \to R_3 - 6R_2}]{} \left[\begin{array}{ccc|c}1 & 5 & -3 & 1\\ 0 & 1 & -1/3 & 1/3\\ 0 & 0 & 0 & 0\end{array}\right]$$

$$\xrightarrow[R_1 \to R_1 - 5R_2]{} \left[\begin{array}{ccc|c}1 & 0 & -4/3 & -2/3\\ 0 & 1 & -1/3 & 1/3\\ 0 & 0 & 0 & 0\end{array}\right]$$

So $\begin{cases} a = -\frac{2}{3} + \frac{4}{3}c \\ b = 1/3 - \frac{1}{3}c \end{cases}$ for any c.

- $\vec{v}_2$ in $Sp(\vec{v}_3, \vec{v}_4)$ $\quad \begin{bmatrix}-1\\-1\\1\end{bmatrix} = a\begin{bmatrix}1\\4\\5\end{bmatrix} + c\begin{bmatrix}1\\0\\-3\end{bmatrix}$

$$\left[\begin{array}{cc|c}1 & 1 & -1\\ 4 & 0 & -1\\ 5 & -3 & 1\end{array}\right] \xrightarrow[\substack{R_2 \to R_2 - 4R_1\\ R_3 \to R_3 - 5R_1}]{} \left[\begin{array}{ccc}1 & 1 & -1\\ 0 & -4 & 3\\ 0 & -8 & 6\end{array}\right] \xrightarrow[\substack{R_2 \to -\frac{1}{4}R_2\\ R_3 \to R_3 - 8R_2}]{} \left[\begin{array}{cc|c}1 & 1 & -1\\ 0 & 1 & -3/4\\ 0 & 0 & 0\end{array}\right]$$

EF

Compatible!

$\vec{v}_3, \vec{v}_4$ not proportional $\rightsquigarrow$ $\{\vec{v}_3, \vec{v}_4\}$ is a basis.

$\boxed{\text{Best spanning sets = Minimal ones (Bases!)}}$

Example: $W = Sp\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}-1\\0\\-7\end{bmatrix}, \begin{bmatrix}2\\7\\0\end{bmatrix}\right) = Sp\left(\begin{bmatrix}1\\0\\7\end{bmatrix}, \begin{bmatrix}0\\1\\-2\end{bmatrix}\right) = Sp\left(\begin{bmatrix}-1\\0\\-7\end{bmatrix}, \begin{bmatrix}2\\7\\0\end{bmatrix}\right)$

Why? $\vec{n} = \begin{bmatrix}1\\0\\7\end{bmatrix} \times \begin{bmatrix}0\\1\\-2\end{bmatrix} = \begin{vmatrix} \check{i} & \check{j} & \check{k} \\ 1 & 0 & 7 \\ 0 & 1 & 2 \end{vmatrix} = -7\check{i} - 2\check{j} + \check{k} = \begin{bmatrix}-7\\-2\\1\end{bmatrix}$

$\rightsquigarrow$ Middle space: plane $\boxed{-7x - 2y + x = 0}$ $(*)$

But Example 1 said W was exactly this plane.

$\rightsquigarrow$ Last space: 2 vectors not proportional & satisfy the same eqn $(*)$

Q: How to get a minimal spanning set for $W = Sp(\vec{v}_1, \ldots, \vec{v}_p)$?

A: Use linear dependencies!

- $7\begin{bmatrix}1\\2\\3\end{bmatrix} + 3\begin{bmatrix}-1\\0\\-7\end{bmatrix} + (-2)\begin{bmatrix}2\\7\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \rightsquigarrow \begin{bmatrix}1\\2\\3\end{bmatrix} = \frac{-3}{7}\begin{bmatrix}-1\\0\\-7\end{bmatrix} + \frac{2}{7}\begin{bmatrix}2\\7\\0\end{bmatrix}$

So we don't need $\begin{bmatrix}1\\2\\3\end{bmatrix}$ to generate W!

$$a\begin{bmatrix}1\\2\\3\end{bmatrix} + b\begin{bmatrix}-1\\0\\-7\end{bmatrix} + c\begin{bmatrix}2\\7\\0\end{bmatrix} = \underbrace{\left(a\left(\tfrac{-3}{7}\right) + b\right)}_{\text{scalars!}}\begin{bmatrix}-1\\0\\-7\end{bmatrix} + \underbrace{\left(a\tfrac{2}{7} + c\right)}_{\text{scalars!}}\begin{bmatrix}2\\7\\0\end{bmatrix}$$

- $\begin{bmatrix}-1\\0\\-7\end{bmatrix}$ & $\begin{bmatrix}2\\7\\0\end{bmatrix}$ are li, so we cannot remove any of them!

$\boxed{\text{ALGORITHM: From Spanning Set to Basis}}$

INPUT: $S = \{\vec{v}_1, \dots, \vec{v}_p\}$ a spanning set of $\mathbb{V}$ (subspace of $\mathbb{R}^n$)

OUTPUT: S' = subset of S that is a basis for $\mathbb{V}$ (= minimal spanning set)

Step ① Is S l.i ?

If YES, then $\boxed{\text{output } S}$

If NO, find a nontrivial solution $(a_1, \dots, a_p)$

to $a_1 \vec{v}_1 + a_2 \vec{v}_2 + \dots + a_p \vec{v}_p = \vec{0}$

Pick smallest index i with $a_i \neq 0$

New $S = \{\vec{v}_1, \dots \vec{v}_{i-1}, \vec{v}_{i+1}, \dots \vec{v}_p\}$

$= S \setminus \{v_i\}$

Step ② Repeat Step ① for New S, ... at some point, we are l.i & we exit.

Example: $S = \left\{ \underset{=\vec{v}_1}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}, \underset{=\vec{v}_2}{\begin{bmatrix} -1 \\ 0 \\ -7 \end{bmatrix}}, \underset{=\vec{v}_3}{\begin{bmatrix} 2 \\ 7 \\ 0 \end{bmatrix}} \right\}$

STEP ①: S is l.d. Write $7 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 0 \\ -7 \end{bmatrix} + (-2) \begin{bmatrix} 2 \\ 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

STEP ② New $S = \left\{ \begin{bmatrix} -1 \\ 0 \\ -7 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 0 \end{bmatrix} \right\}$ l.i $\left(a\vec{v}_2 + b\vec{v}_3 \quad \begin{matrix} 7b = 0 \\ -7a = 0 \end{matrix}, \text{ so } a = b = 0 \right)$

OUTPUT $\{\vec{v}_2, \vec{v}_3\}$

The algorithm gives a new characterization for bases

Proposition: A set $B = \{\vec{v}_1, \dots, \vec{v}_p\}$ is a basis for $\mathbb{W}$ if

(B1) B spans $\mathbb{W}$ & (B2) B is l.i.

⚠ $\{\vec{0}\}$ is l.d. so $\mathbb{W} = \{\vec{0}\}$ has no basis! It is the only subspace of $\mathbb{W}$ without a basis.

Examples (1): $B = \left\{ \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^n$.

Name: Canonical basis / Standard basis

(2) $B' = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is another basis for $\mathbb{R}^3$

Why? • $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = (x-y)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (y-z)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rightsquigarrow$ spans. • Same calculation shows l.i.

(3) $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right\}$ is a basis for $\mathbb{W} = \{ z + 2y - 7x = 0 \}$

Why? • $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 7x-2y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$ so S spans $\mathbb{W}$.

• $a\begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightsquigarrow \begin{matrix} a = 0 \\ b = 0 \\ 7a - 2b = 0 \end{matrix}$ so S is l.i.

A Basis $\mathcal{B}$ gives a new coordinate system for $\mathbb{V}$

Theorem Uniqueness of representation.

Pick a subspace $\mathbb{V} \neq \{\vec{0}\}$ of $\mathbb{R}^n$ with basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_p\}$. Then, any $\vec{v}$ in $\mathbb{V}$ can be represented in a unique way as a linear combination of $\vec{v}_1, \dots, \vec{v}_p$. That is, we can find unique scalars $a_1, \dots, a_p$ so that:

$$\vec{v} = a_1 \vec{v}_1 + \cdots + a_p \vec{v}_p$$

Name: $\begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix} = [\vec{v}]_{\mathcal{B}} =$ coordinates for $\vec{v}$ with respect to the basis $\mathcal{B}$.

Example: $\mathcal{B} = \{e_1, \dots, e_n\}$ gives standard coordinates:

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

So $\left[\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. vs $\left[\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right]_{\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}} = \begin{bmatrix} x-y \\ y-z \\ z \end{bmatrix}$

Theorem Uniqueness of representation.

Pick a subspace $W \neq \{\vec{0}\}$ of $\mathbb{R}^n$ with basis $B = \{\vec{v}_1, \dots, \vec{v}_p\}$.

Then, any $\vec{v}$ in W can be represented in a unique way as a linear combination of $\vec{v}_1, \dots, \vec{v}_p$. That is, we can find unique scalars $a_1, \dots, a_p$ so that:

$$\boxed{\underbrace{\vec{v} = a_1 \vec{v}_1 + \cdots + a_p \vec{v}_p}_{\text{given}}} \qquad (*)$$

- Why is this true?
  - Since B spans W (B1) condition, we can always find $(a_1, \dots, a_p)$ solving $(*)$
  - Uniqueness? Say we have 2 solutions:

$$\vec{v} = a_1 \vec{v}_1 + \cdots + a_p \vec{v}_p = b_1 \vec{v}_1 + \cdots + b_p \vec{v}_p, \quad \text{then}$$

$$a_1 \vec{v}_1 + \cdots + a_p \vec{v}_p - (b_1 \vec{v}_1 + \cdots + b_p \vec{v}_p) = \vec{0}$$

(B2) cond.

Then $(a_1 - b_1)\vec{v}_1 + \cdots + (a_p - b_p)\vec{v}_p = \vec{0}$ forces $a_1 = b_1, \dots, a_p = b_p$ by l.i.