Lecture 17: §3.4 Bases for subspaces
§3.5 Dimension of subspaces

Recall: A **Basis** for $\mathbb{V}$ ($=$ a subspace of $\mathbb{R}^N$), with $\mathbb{V} \neq \{\vec{0}\}$ is a list of vectors $B=\{\vec{v}_1, \ldots, \vec{v}_m\}$ in $\mathbb{V}$ that minimally span $\mathbb{V}$

Alternative: B is a basis for $\mathbb{V}$ if (B1) B spans $\mathbb{V}$ & (B2) B is l.i.

$\boxed{\text{ALGORITHM}}$

- **Input**: A list $S=\{\vec{v}_1, \ldots \vec{v}_p\}$ spanning $\mathbb{V} \neq \{\vec{0}\}$
- **Output**: A — S' included in S that is a basis for $\mathbb{V}$

Step 1 Is S l.i ?

→ If YES, then $\boxed{\text{output } S}$

→ If NO, find a nontrivial solution $(a_1, \ldots a_p)$ to $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_p\vec{v}_p = \vec{0}$

Pick smallest index i with $a_i \neq 0$

$$\text{New } S = \{\vec{v}_1, \ldots \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots \vec{v}_p\}$$
$$= S \setminus \{v_i\}$$

Step 2 Repeat Step 1 for New S, ... At some point, we are li & EXIT.

TODAY:
- 2 more methods to build bases from spanning sets
- All bases for $\mathbb{V}$ have the same size $=$ dimension of $\mathbb{V}$.

Building bases from spanning sets

$S = \{\vec{v}_1, \dots, \vec{v}_p\}$ spans W in $\mathbb{R}^n$ (subspace)

Method ①: View W as the row space of a matrix A (of size $p \times n$)

Then $\begin{bmatrix} \vec{v}_1^{\,t} \\ \vdots \\ \vec{v}_p^{\,t} \end{bmatrix} \xrightarrow{G-J} \begin{bmatrix} \vec{w}_1^{\,t} \\ \vdots \\ \vec{w}_r^{\,t} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ in EF or REF

Output: $B = \{\vec{w}_1, \dots, \vec{w}_r\}$ is a basis for $W = Sp(\vec{v}_1, \dots, \vec{v}_p)$

Example: $W = Sp\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -4 \end{bmatrix} \right)$

Q Why does it work?

① Row space is preserved, so B spans W

② It is l.i. because $a\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} a \\ 2a+b \\ a-3b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ forces $a=0$ & so $b=0$.

(EF / REF are key!)

Method (2) View W as the range of a matrix A of size $n \times p$.

- Find $A \sim A'$ in EF/REF
- Basis $= \{ \vec{v}_i : i$ is a column corresponding to a dependent variable$\}$

  $\hookrightarrow$ size = rank of A'.

Q: Why does this work?

Dimension of W

<u>Def</u>: The <u>dimension</u> of W is the number of elements in any basis for W

Q: How do we know all basis for W have the same size?

<u>Theorem 1</u>: Fix $W \neq \{\vec{0}\}$ a subspace of $\mathbb{R}^n$ & a spanning set $S = \{\vec{w}_1, \cdots \vec{w}_p\}$ for W. Then, any set of $p+1$ or more elements of W is <u>always</u> lin. dep.

<u>Consequence 1</u>: If B & B' are bases for W, then $\text{size}(B) = \text{size}(B')$.

Theorem 1: Fix $W \neq \{\vec{0}\}$ a subspace of $\mathbb{R}^n$ & a spanning set $S = \{\vec{w}_1, \cdots \vec{w}_p\}$ for W. Then, any set of $p+1$ or more elements of W is always lin. dep.

Consequence 2: Fix $W \neq \{\vec{0}\}$ a subspace of $\mathbb{R}^n$ with $\dim W = p$. Then:

① A set of $p+1$ or more vectors in W is linearly dependent.

② Any set of $p-1$ or fewer ______ cannot span W.

③ ______ p linearly indep vectors in W is a basis for W.

④ ______ p vectors in W that spans W ______.

Proof: See lecture notes / textbook

Example $W = Sp\left(\underbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}_{=\vec{v}_1}, \underbrace{\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}}_{=\vec{v}_2}, \underbrace{\begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}}_{=\vec{v}_3}, \underbrace{\begin{bmatrix} -1 \\ -1 \\ -4 \end{bmatrix}}_{=\vec{v}_4} \right)$

Rank and Nullity of a Matrix

A $m \times n$ matrix $\rightsquigarrow$ nullity of $A := \dim(\mathcal{N}(A))$ $\mathcal{N}(A)$ subsp of $\mathbb{R}^n$.

$\rightsquigarrow$ rank of $A := \dim(\mathcal{R}(A))$ $\mathcal{R}(A)$ —— $\mathbb{R}^m$.

Q: How to compute these numbers?

① For nullity: Find $A \sim A'$ REF   nullity = # independent variables

(n − rank A'

② For rank: Use $\mathcal{R}(A) = \mathrm{Row}(A^T)$ & find $A^T \sim A''$ REF

$\mathrm{rank}(A) = $ # non-zero rows of A''. (= # rank A'')

Q: What about our old definition of rank?

Old def: rank (A) = rank (A') = # dep vars if $A \sim A'$ REF

Now: nullity (A) = # indep variables.

Theorem 2. (Rank-Nullity) $\boxed{n = \#\,\text{cols}(A) = \text{rank}(A) + \text{nullity}(A)}$

⚠ We need to match the old def with the new one

Theorem 3: $\text{rank}(A) = \text{rank}(A^T)$, ie the Column & Row Space of A have the same dimension!

Proof: See the lecture notes / textbook. True for REF matrices

(Idea: Show $\text{rank}(A) \leq \text{rank}(A^T)$ & then use $(A^T)^T = A$ to get $\text{rank}(A^T) \leq \text{rank}(A)$)

Two applications

Theorem 4: Fix A of size $m \times n$. The system $A\underline{x} = \underline{b}$ in $\mathbb{R}^n$ is consistent if and only if $\operatorname{rank}(A) = \operatorname{rank}(A|b)$

Consequence: An $n \times n$ matrix A is nonsingular ($\mathcal{N}(A) = \{0\}$) if and only if $\operatorname{rank}(A) = n$