Lecture 17: §3.4 Bases for subspaces
§3.5 Dimension of subspaces

Recall: A **Basis** for $\mathbb{V}$ (= a subspace of $\mathbb{R}^N$), with $\mathbb{V} \neq \{\vec{0}\}$ is a list of vectors $B = \{\vec{v}_1, \ldots, \vec{v}_m\}$ in $\mathbb{V}$ that minimally span $\mathbb{V}$

Alternative: B is a basis for $\mathbb{V}$ if (B1) B spans $\mathbb{V}$ & (B2) B is l.i.

ALGORITHM
- **Input**: A list $S = \{\vec{v}_1, \ldots \vec{v}_p\}$ spanning $\mathbb{V} \neq \{\vec{0}\}$
- **Output**: A — S' included in S that is a basis for $\mathbb{V}$

Step ① Is S l.i ?
- If YES, then $\boxed{\text{output } S}$
- If NO, find a nontrivial solution $(a_1, \ldots a_p)$ to $a_1 \vec{v}_1 + a_2 \vec{v}_2 + \cdots + a_p \vec{v}_p = \vec{0}$

  Pick smallest index i with $a_i \neq 0$

  New $S = \{\vec{v}_1, \ldots \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots \vec{v}_p\}$
  $= S \setminus \{v_i\}$

Step ② Repeat Step ① for New S, ... At some point, we are li & EXIT.

TODAY:
- 2 more methods to build bases from spanning sets
- All bases for $\mathbb{V}$ have the same size = dimension of $\mathbb{V}$.

Building bases from spanning sets

$S = \{\vec{v}_1, \dots, \vec{v}_p\}$ spans $\mathbb{W}$ in $\mathbb{R}^n$ (subspace)

Method ①: View $\mathbb{W}$ as the row space of a matrix A (of size $p \times n$)

Then $\begin{bmatrix} \vec{v}_1^{\,t} \\ \vdots \\ \vec{v}_p^{\,t} \end{bmatrix} \xrightarrow{G-J} \begin{bmatrix} \vec{w}_1^{\,t} \\ \vdots \\ \vec{w}_r^{\,t} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ in EF or REF

Output: $B = \{\vec{w}_1, \dots, \vec{w}_r\}$ is a basis for $\mathbb{W} = Sp(\vec{v}_1, \dots, \vec{v}_p)$

Example: $\mathbb{W} = Sp\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -4 \end{bmatrix} \right)$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 5 \\ 3 & 5 & 6 \\ -1 & -1 & -4 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 + R_1}]{} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 3 \\ 0 & -1 & 3 \\ 0 & 1 & -3 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - R_2 \\ R_4 \to R_4 + R_2 \\ R_2 \to -R_2}]{} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ EF} \rightsquigarrow B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} \right\}$$

$$\downarrow R_1 \to R_1 - 2R_2$$

$$\text{REF} \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} \right\}$$

Q Why does it work?

① Row space is preserved, so B spans $\mathbb{W}$

② It is l.i. because $a \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} a \\ 2a + b \\ a - 3b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ forces $a = 0$ & so $b = 0$.

(EF / REF are key!)

Method ② View $\mathbb{W}$ as the range of a matrix A of size $n \times p$.

- Find $A \sim A'$ in EF/REF
- Basis $= \{\vec{v}_i : i$ is a column corresponding to a dependent variable$\}$
  $\hookrightarrow$ size = rank of A'.

Example: $\mathbb{W} = Sp\left(\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\3\\5\end{bmatrix}, \begin{bmatrix}3\\5\\6\end{bmatrix}, \begin{bmatrix}-1\\-1\\-4\end{bmatrix}\right)$

$$A = \begin{bmatrix}1&2&3&-1\\2&3&5&-1\\1&5&6&-4\end{bmatrix} \xrightarrow[\substack{R_2\to R_2-2R_1\\ R_3\to R_3-R_1}]{} \begin{bmatrix}1&2&3&-1\\0&-1&-1&1\\0&3&3&-3\end{bmatrix} \xrightarrow[\substack{R_3\to R_3+3R_2\\ R_2\to -R_2}]{} \begin{bmatrix}1&2&3&-1\\0&1&1&1\\0&0&0&0\end{bmatrix} \xrightarrow[R_1\to R_1-2R_2]{} \begin{bmatrix}1&0&1&-3\\0&1&1&1\\0&0&0&0\end{bmatrix}$$

dep vars x_1, x_2 REF

So Basis $= \{\vec{v}_1, \vec{v}_2\}$

Q: Why does this work? $Null(A) = \left\{\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} : \begin{cases} x_1 + x_3 - 3x_4 = 0 \\ x_2 + x_3 + x_4 = 0\end{cases}\right\}$

$$\begin{cases} x_1 = -x_3 + 3x_4 \\ x_2 = -x_3 - x_4\end{cases} \rightsquigarrow \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}-x_3+3x_4\\-x_3-x_4\\x_3\\x_4\end{bmatrix} = x_3\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix} + x_4\begin{bmatrix}3\\-1\\0\\1\end{bmatrix}$$

This gives 2 independent relations among $\{\vec{v}_1, \dots, \vec{v}_4\}$:

- $A\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix} = -\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = \vec{0}$
  $\rightsquigarrow \vec{v}_3$ in $Sp(\vec{v}_1, \vec{v}_2)$

&

- $A\begin{bmatrix}3\\-1\\0\\1\end{bmatrix} = 3\vec{v}_1 - \vec{v}_2 + \vec{v}_4 = \vec{0}$
  $\rightsquigarrow \vec{v}_4$ in $Sp(\vec{v}_1, \vec{v}_2)$

Dimension of $\mathbb{W}$

Def: The dimension of $\mathbb{W}$ is the number of elements in any basis for $\mathbb{W}$

Example
- $\mathbb{W} = \{\vec{0}\}$ has no basis, so $\dim\{\vec{0}\} = 0$
- $\mathbb{W} = \mathbb{R}^n$ has dimension n (use standard basis)

Q: How do we know all basis for $\mathbb{W}$ have the same size?

Theorem 1: Fix $\mathbb{W} \neq \{\vec{0}\}$ a subspace of $\mathbb{R}^n$ & a spanning set $S = \{\vec{w}_1, \ldots \vec{w}_p\}$ for $\mathbb{W}$. Then, any set of $p+1$ or more elements of $\mathbb{W}$ is always lin. dep.

Consequence 1: If B & B' are bases for $\mathbb{W}$, then $\text{size}(B) = \text{size}(B')$.

Why? If $\#B = p$ & $\#B' > p$, then B' is ld by Thm 1. This cannot happen since B' is li. So we must have $\#B' \leq \#B$.

By symmetry, if $\#B' < \#B$, then Thm 1 says B is ld, also impossible. $\square$

Example: $\mathbb{W} = \mathbb{R}^3$ $\quad B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ & $B' = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

are both bases for $\mathbb{W}$. Both have size $3 = \dim(\mathbb{R}^3)$.

- This means, by using coordinates for a subspace $\mathbb{W}$ of $\mathbb{R}^n$ with respect to a basis B, we will identify $\mathbb{W}$ with $\mathbb{R}^{\dim \mathbb{W}}$.

Theorem 1: Fix $W \neq \{\vec{0}\}$ a subspace of $\mathbb{R}^n$ & a spanning set $S = \{\vec{w}_1, \dots \vec{w}_p\}$ for W. Then, any set of $p+1$ or more elements of W is *always* lin. dep.

Why? Write $\{\vec{v}_1, \dots, \vec{v}_m\}$ with $m \geq p+1$ for our set. We want to show it's l.d.

Since $\{\vec{w}_1, \dots, \vec{w}_p\}$ spans W, we can find scalars solving:

$$\begin{cases} \vec{v}_1 = a_{11}\vec{w}_1 + \dots + a_{1p}\vec{w}_p \\ \vdots \\ \vec{v}_m = a_{m1}\vec{w}_1 + \dots + a_{mp}\vec{w}_p \end{cases} \rightsquigarrow \underbrace{\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_m \end{bmatrix}}_{n \times m} = \underbrace{\begin{bmatrix} \vec{w}_1 & \cdots & \vec{w}_p \end{bmatrix}}_{n \times p} \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & & \vdots \\ a_{1p} & & a_{mp} \end{bmatrix}}_{p \times m}$$

GOAL: Show $x_1\vec{v}_1 + \dots + x_m\vec{v}_m = \vec{0}$ has a nontrivial soln $(x_1, \dots, x_m)$

Rewrite it as
$$\vec{0} = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_m \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \underbrace{\begin{bmatrix} \vec{w}_1 & \cdots & \vec{w}_p \end{bmatrix}}_{n \times p} \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & & \vdots \\ a_{1p} & & a_{mp} \end{bmatrix}}_{A \text{ of size } p \times m} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$$

Since $p < m$, $A \sim A'$ with $\operatorname{rank}(A') \leq$ rows of $A = p < m$, so the system $A \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \underset{\text{in } \mathbb{R}^p}{\vec{0}}$ has a nontrivial solution (we have indep vars!)

So
$$\begin{bmatrix} \vec{w}_1 & \cdots & \vec{w}_p \end{bmatrix} A \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \underbrace{\begin{bmatrix} \vec{w}_1, & \cdots & \vec{w}_p \end{bmatrix}}_{n \times p} \underbrace{\vec{0}}_{p \times 1} = \vec{0} \text{ in } \mathbb{R}^n.$$

Conclude: $\{\vec{v}_1, \dots, \vec{v}_m\}$ is l.d.

Consequence 2: Fix $W \neq \{\vec{0}\}$ a subspace of $\mathbb{R}^n$ with $\dim W = p$. Then:

1. A set of $p+1$ or more vectors in W is linearly dependent.
2. Any set of $p-1$ or fewer ______________ cannot span W.
3. ______________ p linearly indep vectors in W is a basis for W.
4. ______________ p vectors in W that spans W ______________.

Proof: See lecture notes / textbook

Example

$$W = \text{Sp}\left(\overset{=\vec{v}_1}{\begin{bmatrix}1\\2\\1\end{bmatrix}}, \overset{=\vec{v}_2}{\begin{bmatrix}2\\3\\5\end{bmatrix}}, \overset{=\vec{v}_3}{\begin{bmatrix}3\\5\\6\end{bmatrix}}, \overset{=\vec{v}_4}{\begin{bmatrix}-1\\-1\\-4\end{bmatrix}} \right)$$

We computed a basis for W earlier: $B = \left\{ \begin{bmatrix}1\\0\\7\end{bmatrix}, \begin{bmatrix}0\\1\\-3\end{bmatrix} \right\}$, so $\dim W = 2$.

- By Consequence 2 ① $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is l.d.
- ②: $\{\vec{v}_1\}$ cannot span W. Same for each $\{\vec{v}_2\}$, $\{\vec{v}_3\}$ or $\{\vec{v}_4\}$.
- ③ $B_{12} = \{\vec{v}_1, \vec{v}_2\}$ is l.i

$$\begin{bmatrix}1 & 2\\2 & 3\\1 & 5\end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix}1 & 2\\0 & -1\\0 & 3\end{bmatrix} \xrightarrow[R_2 \to -R_2]{R_3 \to R_3 + 3R_2} \begin{bmatrix}1 & 2\\0 & 1\\0 & 0\end{bmatrix}$$

no indep vars so l.i

So B_{12} is a basis for W

Similarly $\{\vec{v}_1, \vec{v}_3\}$, $\{\vec{v}_1, \vec{v}_4\}$, $\{\vec{v}_2, \vec{v}_3\}$, $\{\vec{v}_2, \vec{v}_4\}$, $\{\vec{v}_3, \vec{v}_4\}$ are li of size 2, so they are all bases for W.

Rank and Nullity of a Matrix

A $m \times n$ matrix $\rightsquigarrow$ nullity of $A := \dim(\mathcal{N}(A))$ $\qquad$ $\mathcal{N}(A)$ subsp of $\mathbb{R}^n$.

$\rightsquigarrow$ rank of $A := \dim(\mathcal{R}(A))$ $\qquad$ $\mathcal{R}(A)$ — $\mathbb{R}^m$.

Q: How to compute these numbers?

① <u>For nullity</u>: Find $A \sim A'$ REF $\qquad$ nullity $=$ # independent variables $(n - \text{rank } A')$

<u>Example</u>: $A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 3 & 5 & -1 \\ 1 & 5 & 6 & -4 \end{bmatrix} \xrightarrow{G\text{-}J} \underbrace{\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{REF}} = A'$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 - x_4 \\ -x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$\mathcal{N}(A) = \text{Sp}\left(\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right)$ The vectors are li, so $\dim \mathcal{N}(A) = 2$.
(0's in complementary positions!)

② <u>For rank</u>: Use $\mathcal{R}(A) = \text{Row}(A^T)$ & find $A^T \sim A''$ REF

rank(A) $=$ # non-zero rows of A'' $(= \# \text{ rank } A'')$.

<u>Example</u>: $A^T = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 5 \\ 3 & 5 & 6 \\ -1 & -1 & -4 \end{bmatrix} \xrightarrow{G\text{-}J} \underbrace{\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{REF}} = A''$ $\qquad$ rank$(A'') = 2$

So rank$(A) = 2$

Q: What about our old definition of rank?

Old def: rank(A) = rank(A') = # dep vars if $A \sim A'$ REF

Now: nullity (A) = # indep variables.

Theorem 2. $\boxed{n = \#\text{cols}(A) = \text{rank}(A) + \text{nullity}(A)}$
(Rank-Nullity)

⚠ We need to match the old def with the new one

- New def rank(A) = dim$(R(A))$ = dim$(\text{Rows}(A^T))$

  Use $A^T \sim A''$ REF rank(A'') = rank(A)

- Old def: use $A^T \sim A''$ REF would give rank(A^T)

Key result solves this issue:

Theorem 3: rank(A) = rank(A^T), ie the Column & Row Space of A have the same dimension!

Proof: See the lecture notes / textbook. True for REF matrices.

(Idea: Show rank$(A) \leq$ rank(A^T) & then use $(A^T)^T = A$ to get rank$(A^T) \leq$ rank(A))

$\boxed{\text{Two applications}}$

Theorem 4: Fix A of size $m \times n$. The system $A\underline{x} = \underline{b}$ in $\mathbb{R}^n$ is consistent if and only if $\text{rank}(A) = \text{rank}(A|b)$

Why? Consistent means $\underline{b}$ in $Sp(\text{col}_1(A), \ldots, \text{col}_n(A)) = \mathcal{R}(A)$

So $\dim \mathcal{R}(A) = \dim \mathcal{R}(A|b)$. These dimensions are the ranks.

Consequence: An $n \times n$ matrix A is nonsingular ($\mathcal{N}(A) = \{0\}$) if and only if $\text{rank}(A) = n$

Why? Nonsingular $\equiv$ $\text{nullity}(A) = 0$ $\equiv$ $\text{rank}(A) = n - \text{nullity}(A) = n$.

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix}$ nullity $= ?$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow[R_2 \to R_2 - R_1]{} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix} \xrightarrow[R_2/2]{} \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}}_{\text{REF}}$$

x_1, x_2 dep
x_3 indep

$\mathcal{N}(A) = Sp\left(\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right)$, so nullity $= 1$ $\rightsquigarrow$ $\text{rank}(A) = 3 - 1 = 2$.

- $\text{rank}(A) = \dim \mathcal{R}(A)$ & $\mathcal{R}(A)$ is a subset of $\mathbb{R}^2$ of dim 2. So $\boxed{\mathcal{R}(A) = \mathbb{R}^2}$
- $\text{rank}(A^T) = \text{rank}(A) = 2$ $\quad B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\text{Rows}(A)$.