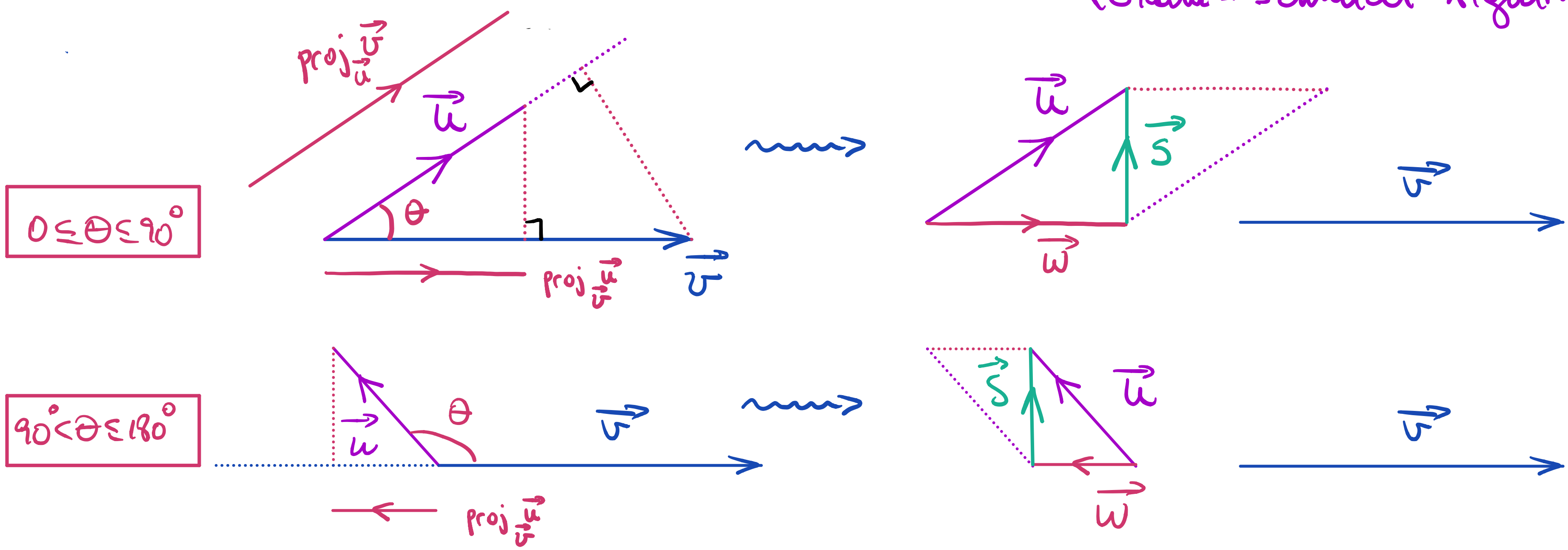
Lecture 18: §3.6 Orthogonal bases for subspaces

TODAY: Discuss inner products (generalizing dot products in $\mathbb{R}^n$) & find bases that are well-behaved with respect to these inner products.
(Gram-Schmidt Algorithm)



$$\boxed{\vec{u} = \underbrace{\mathrm{proj}_{\vec{v}}\vec{u}}_{\vec{w}} + \underbrace{(u - \mathrm{proj}_{\vec{v}}\vec{u})}_{\vec{s}}}$$

where $\vec{w} \,/\!/\, \vec{v}$
$\vec{s} \perp \vec{v}$

Key: $\vec{w}$ & $\vec{s}$ are the only vectors with these properties. (L11)

Basis: $\{\vec{v}, \vec{u}\} \rightsquigarrow \{\vec{v}, \vec{s}\}$ $\quad \vec{v} \perp \vec{s}$ (like e_1 & e_2)

Inner Products of Subspaces $\mathbb{V}$ of $\mathbb{R}^n$

<u>Def</u>: An <u>inner product</u> for $\mathbb{V}$ is a function $\langle\, ,\, \rangle : V \times V \longrightarrow \mathbb{R}$ assigning a number $\langle \vec{u}, \vec{v} \rangle$ to each pair of vectors $\vec{u}, \vec{v}$ in $\mathbb{V}$ & satisfying the following properties:

① $\langle \vec{u}, \vec{u} \rangle \geq 0$ for all $\vec{u}$; $\langle \vec{u}, \vec{u} \rangle = 0$ if & only if $\vec{u} = \vec{0}$

② $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$ (Symmetric)

③ $\langle \alpha \vec{u}, \vec{v} \rangle = \alpha \langle \vec{u}, \vec{v} \rangle = \langle \vec{u}, \alpha \vec{v} \rangle$ for any scalar α.

④ $\langle \vec{u} + \vec{w}, \vec{v} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{w}, \vec{v} \rangle$

$\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$

Example 2: Pick Q $n \times n$ symmetric matrix of rank n

(symmetric + invertible Eg $Q = I_n$)

Check the 4 properties for an inner product:

3 $\langle \alpha \vec{u}, \vec{v} \rangle = \alpha \langle \vec{u}, \vec{v} \rangle = \langle \vec{u}, \alpha \vec{v} \rangle$ for any scalar α

4 $\langle \vec{u} + \vec{w}, \vec{v} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{w}, \vec{v} \rangle$ & $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$

2 $\langle \vec{u}, \vec{v} \rangle \overset{?}{=} \langle \vec{v}, \vec{u} \rangle$ (Symmetric)

1 $\langle \vec{u}, \vec{u} \rangle \geq 0$ for all $\vec{u}$; $\langle \vec{u}, \vec{u} \rangle = 0$ if & only if $\vec{u} = \vec{0}$.

Orthogonal bases for $\mathbb{R}^n$

Fix $\langle\, ,\, \rangle$ = dot product

Def: $\vec{u}$ & $\vec{v}$ in $\mathbb{R}^n$ are **orthogonal** if

Def: A **set** of vectors $S = \{\vec{v}_1, \dots, \vec{v}_p\}$ in $\mathbb{R}^n$ is **orthogonal** if

Q: Why do we care?

Theorem: Orthogonal set **NOT** containing $\vec{0}$ are **ALWAYS l.i.**

Fix $W \neq \{\vec{0}\}$ a subspace of $\mathbb{R}^n$ with basis $B = \{\vec{w}_1, \dots, \vec{w}_p\}$

Def We say B is an orthogonal basis if

____________________ orthonormal basis if

Main advantage of orthonormal bases: fast to write down coordinates!

Theorem: Fix $B = \{\vec{v}_1, \ldots, \vec{v}_p\}$ orthonormal basis for $W \neq \{\vec{0}\}$ in $\mathbb{R}^n$.

Then, for each $\vec{v}$ in W $\quad [\vec{v}]_B = \begin{bmatrix} \vec{v} \cdot \vec{v}_1 \\ \vdots \\ \vec{v} \cdot \vec{v}_p \end{bmatrix}$ in $\mathbb{R}^p$,

meaning $\vec{v} = \underbrace{(\vec{v} \cdot \vec{v}_1)}\vec{v}_1 + \underbrace{(\vec{v} \cdot \vec{v}_2)}\vec{v}_2 + \cdots + \underbrace{(\vec{v} \cdot \vec{v}_p)}\vec{v}_p$.

$\leftarrow$ scalars

Gram-Schmidt Algorithm

Build orthogonal basis from any basis

- **INPUT:** $B = \{\vec{w}_1, \dots, \vec{w}_p\}$ basis for subspace $\mathbb{W}$ of $\mathbb{R}^n$
- **OUTPUT:** $B' = \{\vec{u}_1, \dots, \vec{u}_p\}$ orthogonal basis for $\mathbb{W}$.

Routine: $\vec{u}_1 = \vec{w}_1$

$$\vec{u}_2 = \vec{w}_2 - \operatorname{proj}_{\vec{u}_1} \vec{w}_2 = \vec{w}_2 - \frac{\vec{u}_1 \cdot \vec{w}_2}{\|\vec{u}_1\|^2} \vec{u}_1$$

$$u_3 = \vec{w}_3 - \operatorname{proj}_{\vec{u}_1} \vec{w}_3 - \operatorname{proj}_{\vec{u}_2} \vec{w}_3 = \vec{w}_3 - \frac{\vec{u}_1 \cdot \vec{w}_3}{\|\vec{u}_1\|^2} \vec{u}_1 - \frac{\vec{u}_2 \cdot \vec{w}_3}{\|\vec{u}_2\|^2} \vec{u}_2$$

$\vdots$

$$\vec{u}_j = \vec{w}_j - \frac{\vec{u}_1 \cdot \vec{w}_j}{\|\vec{u}_1\|^2}\vec{u}_1 - \frac{\vec{u}_2 \cdot \vec{w}_j}{\|\vec{u}_2\|^2}\vec{u}_2 - \cdots - \frac{\vec{u}_{j-1} \cdot \vec{w}_j}{\|\vec{u}_{j-1}\|^2}\vec{u}_{j-1}$$

scalars

Q: Why these scalars?

$\boxed{\text{Example}}$ $V = \mathbb{R}^3$

INPUT: $\left\{ \underset{\vec{w}_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{\vec{w}_2}{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}}, \underset{\vec{w}_3}{\begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}} \right\}$

OUTPUT $= \{\vec{u}_1, \vec{u}_2, \vec{u}_3\} =$