Lecture 19: §3.7 Linear Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ I

GOAL: Study functions between (subspaces of) $\mathbb{R}^n$ & $\mathbb{R}^m$ that respect the vector space structure on both sides (addition of vectors & scalar multiplication)

- Same ideas will work when we move from $\mathbb{R}^n$ to abstract vector spaces
- Meta example: multiplication by a fixed matrix A of size $m \times n$

Example

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}$$
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longmapsto x_1 + 5x_3$$

Observation: $f:\mathbb{R}^3 \longrightarrow \mathbb{R}$ $\quad f\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = x_1+5x_3$

We can restrict to a line or a plane through $(0,0,0)$ and get a new functions $f_1 = f|_L : L \longrightarrow \mathbb{R}$ & $f_2 = f|_{\text{plane}} : \text{plane} \longrightarrow \mathbb{R}$

How?

<u>Example</u> Build $G: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ linear using 2 linear functions

$F_1: \mathbb{R}^3 \longrightarrow \mathbb{R}$ & $F_2: \mathbb{R}^3 \longrightarrow \mathbb{R}$ $\rightsquigarrow G(\vec{v}) = \begin{bmatrix} F_1(\vec{v}) \\ F_2(\vec{v}) \end{bmatrix}$

(1st coord of G) (2nd coordinate of G)

Ex: $F_1(\vec{x}) = x_1 + 5x_3 =$

$F_2(\vec{x}) = 3x_1 - 7x_2 + 8x_3 =$

The values of G at the standard basis of $\mathbb{R}^3$ determine G!

$G\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\3\end{bmatrix} = \mathrm{col}_1(A)$, $G\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\-7\end{bmatrix} = \mathrm{col}_2(A)$, $G\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\8\end{bmatrix} = \mathrm{col}_3(A)$

Then $G\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) =$

So we recover $G\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)$ from its values at the standard basis. The same will be true for any other choice of basis for $\mathbb{R}^3$.

$$G\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 5x_3 \\ 3x_1 - 7x_2 + 8x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 5 \\ 3 & -7 & 8 \end{bmatrix}}_{= A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Q1: What is the image of G?

A

Q2: What vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ map to $\vec{0}$ in $\mathbb{R}^2$ under G?

A

Summary of examples:

① All linear maps $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ are determined by multiplication by an $m \times n$ matrix A $(f(\vec{v}) = A\vec{v})$.

② Image of the map f = Range of A (Column Space)

③ Vectors $\vec{v}$ with $f(\vec{v}) = \vec{0}$ = Null Space of A.

These 3 conditions are true for any linear map $f: \mathbb{R}^n \to \mathbb{R}^m$

General definition of linear maps = linear Transformations

Def: A function $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation if

(T1) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ in $\mathbb{R}^m$ for all $\vec{u}, \vec{v}$ in $\mathbb{R}^n$

&

(T2) $T(\alpha \vec{u}) = \alpha T(\vec{u})$ in $\mathbb{R}^m$ for all $\vec{u}$ in $\mathbb{R}^n$, α in $\mathbb{R}$

$$\boxed{T:\mathbb{R}^n \longrightarrow \mathbb{R}^m \quad T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \quad \& \quad T(\alpha \vec{u}) = \alpha T(\vec{u})}$$

<u>Non-example</u> 1 $F:\mathbb{R}^2 \longrightarrow \mathbb{R}^3$ $F(\underline{x}) = \begin{bmatrix} x_1 - x_2 + 1 \\ x_2 \\ 2x_1 + x_2 \end{bmatrix}$

<u>Non-example</u> 2 : $F:\mathbb{R} \longrightarrow \mathbb{R}$ $F(x) = e^x$

Necessary condition for being linear

$$\boxed{T:\mathbb{R}^n \longrightarrow \mathbb{R}^m \quad T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \quad \& \quad T(\alpha\vec{u}) = \alpha T(\vec{u})}$$

Special cases: $m=1$ Q: Formulas for linear transformations?

Proposition 1: $T:\mathbb{R} \longrightarrow \mathbb{R}$ is linear if, and only if $T(x) = ax$ for some fixed constant a in $\mathbb{R}$. Moreover, $a = T(1)$.

Proof

Examples ① $T:\mathbb{R} \longrightarrow \mathbb{R}$ $T(x) = 3x$

② $T:\mathbb{R} \longrightarrow \mathbb{R}$ $T(x) = \sin x$

③ $T:\mathbb{R} \longrightarrow \mathbb{R}$ $T(x) = 3x - 4$

$$\boxed{T:\mathbb{R}^n \longrightarrow \mathbb{R}^m \quad T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \quad \& \quad T(\alpha \vec{u}) = \alpha T(\vec{u})}$$

Proposition 2: $T:\mathbb{R}^n \longrightarrow \mathbb{R}$ is a linear transformation if, and only if, $T(\vec{x}) = \vec{u}^T \vec{x}$ for some vector $\vec{u}$ in $\mathbb{R}^n$. Moreover, $\vec{u} = \begin{bmatrix} T(\vec{e}_1) \\ T(\vec{e}_2) \\ \vdots \\ T(\vec{e}_n) \end{bmatrix}$

General form of $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ linear transf

<u>Theorem</u>. Every linear transf $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ has the form

$$T\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$ for some $m \times n$ matrix A.

Moreover $A = \left[\, T(\vec{e}_1) \;\cdots\; T(\vec{e}_n) \,\right]$.

Exercise 1 Find a linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ with

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\4\\3\end{bmatrix} \quad \& \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\\5\end{bmatrix}$$

Exercise 2: Same question but $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\4\\3\end{bmatrix}$ & $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}1\\0\\5\end{bmatrix}$