Lecture 19: §3.7 Linear Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ I

GOAL: Study functions between (subspaces of) $\mathbb{R}^n$ & $\mathbb{R}^m$ that respect the vector space structure on both sides (addition of vectors & scalar multiplication)

- Same ideas will work when we move from $\mathbb{R}^n$ to abstract vector spaces
- Meta example: multiplication by a fixed matrix A of size $m \times n$

Example $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ $\quad (n=3,\ m=1)$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longmapsto x_1 + 5x_3 = x_1 + 0 \cdot x_2 + 5x_3 = \begin{bmatrix} \overset{f(e_1)}{\overset{\parallel}{1}} & \overset{f(e_2)}{\overset{\parallel}{0}} & \overset{f(e_3)}{\overset{\parallel}{5}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

linear expression in x_1, x_2, x_3.

$$f\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = 1+0 = 1 \qquad f\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = 0+0 = 0 \qquad f\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = 0 + 5 \cdot 1 = 5$$

Q: What does linear mean? Sums & scalar mult are respected by f.

(1) $$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = f\left(\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{bmatrix}\right) = (x_1+y_1) + 5(x_3+y_3) = \underbrace{x_1+5x_3}_{} + \underbrace{y_1+5y_3}_{}$$
$$= f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) + f\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right)$$

(2) $$f\left(\alpha \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = f\left(\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{bmatrix}\right) = \alpha x_1 + 5(\alpha x_3) = \alpha(x_1 + 5x_3) = \alpha f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)$$

Observation: $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ $\quad f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = x_1 + 5x_3$

We can restrict to a line or a plane through $(0,0,0)$ and get 2 new functions $f_1 = f|_L : L \longrightarrow \mathbb{R}$ & $f_2 = f|_{\text{plane}}$ plane $\longrightarrow \mathbb{R}$

How? • L line is the linear span of 1 vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \neq \vec{0}$.

So $f_1\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = f_1(t\vec{v}) = t\, f_1(\vec{v}) = t(v_1 - 5v_3)$

Ex $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ $\quad f_1(t\vec{v}) = t(1 - 5\cdot 2) = -9t$

This looks like $f_1 : \mathbb{R} \longrightarrow \mathbb{R}$, $t \longmapsto -9t$ $\quad$ Image of $f = \mathbb{R}$. Injective

• Plane $= Sp(\vec{v}, \vec{w})$ with $\vec{v}, \vec{w}$ not proportional.

So $f_2\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = f_2(a\vec{v} + b\vec{w}) = a\, f_2(\vec{v}) + b\, f_2(\vec{w})$

Ex: $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$ $\quad f_2(a\vec{v} + b\vec{w}) = a(1-10) + b(-5-5) = -9a - 10b$.

This looks like $f_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ $\quad (a,b) \longmapsto -9a - 10b$.

Example Build $G: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ linear using 2 linear functions

$F_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}$ & $F_2 : \mathbb{R}^3 \longrightarrow \mathbb{R}$ $\rightsquigarrow G(\vec{v}) = \begin{bmatrix} F_1(\vec{v}) \\ F_2(\vec{v}) \end{bmatrix}$

(1st coord of G) (2nd coordinate of G)

Ex: $F_1(\vec{x}) = x_1 + 5x_3 = [1\ 0\ 5] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$F_2(\vec{x}) = 3x_1 - 7x_2 + 8x_3 = [3\ -7\ 8] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\rightsquigarrow G\Big(\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\text{in } \mathbb{R}^3}\Big) = \begin{bmatrix} x_1 + 5x_3 \\ 3x_1 - 7x_2 + 8x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 5 \\ 3 & -7 & 8 \end{bmatrix}}_{\substack{=A \\ 2\times 3}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{3\times 1} \text{ in } \mathbb{R}^2$$

Conclusion: $G: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ becomes matrix multiplication by a fixed 2×3 matrix A

- # rows of $A = 2$ = dim of target space $\mathbb{R}^2$
- # cols of $A = 3$ = —— source —— $\mathbb{R}^3$

Crucial Obs: What are the images of the standard basis vectors from $\mathbb{R}^3$?

$G\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = col_1(A)$, $G\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -7 \end{bmatrix} = col_2(A)$, $G\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 8 \end{bmatrix} = col_3(A)$

$\rightsquigarrow$ We get the columns of A! So A determines G!

The values of G at the standard basis of $\mathbb{R}^3$ determine G!

$$G\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\3\end{bmatrix} = \mathrm{col}_1(A),\quad G\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\-7\end{bmatrix} = \mathrm{col}_2(A),\quad G\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\8\end{bmatrix} = \mathrm{col}_3(A)$$

Then $G\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = G\left(x_1\begin{bmatrix}1\\0\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\\0\end{bmatrix} + x_3\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$

$$\overset{G \text{ linear}}{=} x_1 G\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) + x_2 G\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) + x_3 G\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$

$$= x_1\begin{bmatrix}1\\3\end{bmatrix} + x_2\begin{bmatrix}0\\-7\end{bmatrix} + x_3\begin{bmatrix}5\\8\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0 & 5\\3 & -7 & 8\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_1 + 5x_3\\3x_1 - 7x_2 + 8x_3\end{bmatrix}$$

So we recover $G\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)$ from its values at the standard basis. The same will be true for any other choice of basis for $\mathbb{R}^3$.

$$G\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 5x_3 \\ 3x_1 - 7x_2 + 8x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 5 \\ 3 & -7 & 8 \end{bmatrix}}_{=A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Q1: What is the image of G?

A Vectors $\vec{w}$ in $\mathbb{R}^2$ of the form $\vec{w} = A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ for $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathbb{R}^3$

This is $R(A) = Sp\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -7 \end{bmatrix}, \begin{bmatrix} 5 \\ 8 \end{bmatrix}\right)$ (Image = Range of A)

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & w_1 \\ 3 & -7 & 8 & w_2 \end{array}\right] \xrightarrow[R_2 \to R_2 - 3R_1]{} \left[\begin{array}{ccc|c} 1 & 0 & 5 & w_1 \\ 0 & -7 & -7 & w_2 - 3w_1 \end{array}\right] \xrightarrow[R_2 \to -\frac{R_2}{7}]{} \left[\begin{array}{ccc|c} 1 & 0 & 5 & w_1 \\ 0 & 1 & 1 & -\frac{w_2}{7} + \frac{3}{7}w_1 \end{array}\right]$$

REF

Always compatible! So $\boxed{R(A) = \mathbb{R}^2}$ dim 2

Q2: What vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ map to $\vec{0}$ in $\mathbb{R}^2$ under G?

(Rank-Nullity gives $\dim N(A) = 1$)

A Need to solve $A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow$ This is $N(A)$ = Nullspace of A

$$\begin{bmatrix} 1 & 0 & 5 \\ 3 & -7 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 1 \end{bmatrix}$$

x_3 indep $\quad \begin{cases} x_1 = -5x_3 \\ x_2 = x_3 \end{cases}$

$N(A) = Sp\left(\begin{bmatrix} -5 \\ 1 \end{bmatrix}\right)$ dim 1

Check $G\begin{pmatrix} -5 \\ 1 \end{pmatrix} = \begin{bmatrix} -5 + 5\cdot 1 \\ 3(-5) - 7\cdot 1 + 8\cdot 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Summary of examples:

① All linear maps $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ are determined by multiplication by an $m \times n$ matrix A $\left(f(\vec{v}) = A\vec{v} \right)$.

② Image of the map f = Range of A (Column Space)

③ Vectors $\vec{v}$ with $f(\vec{v}) = \vec{0}$ = Null Space of A.

These 3 conditions are true for any linear map $f: \mathbb{R}^n \to \mathbb{R}^m$

$\boxed{\text{General definition of linear maps = linear Transformations}}$

Def: A function $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation if

(T1) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ in $\mathbb{R}^m$ for all $\vec{u}, \vec{v}$ in $\mathbb{R}^n$

&

(T2) $T(\alpha \vec{u}) = \alpha T(\vec{u})$ in $\mathbb{R}^m$ for all $\vec{u}$ in $\mathbb{R}^n$, α in $\mathbb{R}$

$$\boxed{T:\mathbb{R}^n \longrightarrow \mathbb{R}^m \quad T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \quad \& \quad T(\alpha \vec{u}) = \alpha T(\vec{u})}$$

Non-example ① $F:\mathbb{R}^2 \longrightarrow \mathbb{R}^3$ $\quad F(\underline{x}) = \begin{bmatrix} x_1 - x_2 + 1 \\ x_2 \\ 2x_1 + x_2 \end{bmatrix}$ problem! NOT linear

$$F\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\0\\2\end{bmatrix}, \quad F\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\\1\end{bmatrix} \quad \& \quad F\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\1\\3\end{bmatrix}$$

BUT $\begin{bmatrix}2\\0\\2\end{bmatrix} + \begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}2\\1\\3\end{bmatrix} \neq \begin{bmatrix}1\\1\\3\end{bmatrix}$. $\rightsquigarrow$ (T1) fails

Non-example ②: $F:\mathbb{R} \longrightarrow \mathbb{R}$ $\quad F(x) = e^x$ is NOT linear

$F(0) = 1, \quad F(1) = e \quad F(0) + F(1) = 1 + e \neq F(0+1)$

$F(2) = F(2\cdot 1) = e^2 \neq 2e = 2F(1)$

$\rightsquigarrow$ (T1) & (T2) fails.

Necessary condition for being linear $F(\vec{0}) = \vec{0}$.

Why? $F(\vec{0}) = F(\vec{0} + \vec{0}) = F(\vec{0}) + F(\vec{0}) = 2F(\vec{0}) \rightsquigarrow F(\vec{0}) = \vec{0}$

$$\boxed{T:\mathbb{R}^n \longrightarrow \mathbb{R}^m \quad T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \quad \& \quad T(\alpha\vec{u}) = \alpha T(\vec{u})}$$

Special cases: $m=1$ Q: Formulas for linear transformations?

Proposition 1: $T:\mathbb{R} \longrightarrow \mathbb{R}$ is linear if, and only if $T(x) = ax$ for some fixed constant a in $\mathbb{R}$. Moreover, $a = T(1)$.

Proof $T(x) = ax$ is clearly linear, since

(T1) $T(x+y) = a(x+y) = ax + ay = T(x) + T(y)$ ✓

(T2) $T(\alpha x) = a(\alpha x) = \alpha(ax) = \alpha T(x)$ ✓

If T is linear, then $T(x) = T(\underbrace{x}_{\text{scalar!}} \cdot 1) \overset{T\text{ linear}}{=} x\,\underbrace{T(1)}_{=a} = ax$.

Examples ① $T:\mathbb{R} \longrightarrow \mathbb{R}$ $T(x) = 3x$ linear

② $T:\mathbb{R} \longrightarrow \mathbb{R}$ $T(x) = \sin x$ NOT linear

③ $T:\mathbb{R} \longrightarrow \mathbb{R}$ $T(x) = 3x - 4$ NOT linear

$$\boxed{T:\mathbb{R}^n \longrightarrow \mathbb{R}^m \quad T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \quad \& \quad T(\alpha\vec{u}) = \alpha T(\vec{u})}$$

Proposition 2: $T:\mathbb{R}^n \longrightarrow \mathbb{R}$ is a linear transformation if, and only if, $T(\vec{x}) = \vec{u}^T\vec{x}$ for some vector $\vec{u}$ in $\mathbb{R}^n$. Moreover, $\vec{u} = \begin{bmatrix} T(\vec{e}_1) \\ T(\vec{e}_2) \\ \vdots \\ T(\vec{e}_n) \end{bmatrix}$

Why? • If $T(\vec{x}) = \vec{u}^T\vec{x} = \vec{u}\cdot\vec{x}$ (dot product with a fixed $\vec{u}$)

We know this is linear (Properties of dot product)

• Conversely, assume T is linear. Then

$$T\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = T(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n)$$

$$\overset{(T1)}{=} T(x_1\vec{e}_1) + T(x_2\vec{e}_2 + \cdots + x_n\vec{e}_n)$$

$$= T(x_1\vec{e}_1) + T(x_2\vec{e}_2) + \cdots + T(x_n\vec{e}_n)$$

$$\overset{(T2)}{=} x_1T(\vec{e}_1) + x_2T(\vec{e}_2) + \cdots + x_nT(\vec{e}_n)$$

$$= \underbrace{\boxed{\begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix}}}_{=\vec{u}^T} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

General form of $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ linear transf

We build T from m linear maps

$T_1 : \mathbb{R}^n \longrightarrow \mathbb{R}$ (word 1 of T)

$T_2 : \mathbb{R}^n \longrightarrow \mathbb{R}$ (word 2 of T)

$\vdots$

$T_m : \mathbb{R}^n \longrightarrow \mathbb{R}$ (word m of T)

By Proposition 2: $T_1(\vec{x}) = \vec{v}_1^{\,T} \vec{x}$ for some vector $\vec{v}_1$ in $\mathbb{R}^n$

$\vdots$

$T_m(\vec{x}) = \vec{v}_m^{\,T} \vec{x}$ ———————— $\vec{v}_m$ in $\mathbb{R}^n$

So
$$T(\vec{x}) = \begin{bmatrix} T_1(\vec{x}) \\ \vdots \\ T_m(\vec{x}) \end{bmatrix} = \begin{bmatrix} \vec{v}_1^{\,T} \vec{x} \\ \vdots \\ \vec{v}_m^{\,T} \vec{x} \end{bmatrix} = \underbrace{\begin{bmatrix} \vec{v}_1^{\,T} \\ \vdots \\ \vec{v}_m^{\,T} \end{bmatrix}}_{= A \text{ of size } m \times n} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Theorem. Every linear transf $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ has the form
$$T\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$
for some $m \times n$ matrix A.

Moreover $A = [\, T(\vec{e}_1) \; \cdots \; T(\vec{e}_n) \,]$. (columns are the image of the standard basis for $\mathbb{R}^n$ under T)

Exercise 1 Find a linear transformation $T:\mathbb{R}^2 \longrightarrow \mathbb{R}^3$ with

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\4\\3\end{bmatrix} \quad \& \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\\5\end{bmatrix}$$

Solution $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = A\begin{bmatrix}x\\y\end{bmatrix}$ $\quad A = \left[T(\vec{e}_1)\ T(\vec{e}_2)\right] = \begin{bmatrix}1 & 1\\4 & 0\\3 & 5\end{bmatrix}$ 3×2

So $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}1 & 1\\4 & 0\\3 & 5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x+y\\4x\\3x+5y\end{bmatrix}$ (UNIQUE CHOICE)

Exercise 2: Same question but $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\4\\3\end{bmatrix}$ & $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}1\\0\\5\end{bmatrix}$

Solution: Need to compute $T(\vec{e}_1)$ & $T(\vec{e}_2)$, so write $\vec{e}_1$ & $\vec{e}_2$ in terms of $\begin{bmatrix}1\\1\end{bmatrix}$ & $\begin{bmatrix}1\\2\end{bmatrix}$ & use linear properties of T.

$$\begin{bmatrix}1\\0\end{bmatrix} = 2\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}1\\2\end{bmatrix} \rightsquigarrow T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = 2T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) - T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = 2\begin{bmatrix}1\\4\\3\end{bmatrix} - \begin{bmatrix}1\\0\\5\end{bmatrix} = \begin{bmatrix}1\\8\\1\end{bmatrix}$$

$$\begin{bmatrix}0\\1\end{bmatrix} = -\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix} \rightsquigarrow T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = -T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) + T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = -\begin{bmatrix}1\\4\\3\end{bmatrix} + \begin{bmatrix}1\\0\\5\end{bmatrix} = \begin{bmatrix}0\\-4\\2\end{bmatrix}$$

So $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \left[T(\vec{e}_1)\ T(\vec{e}_2)\right]\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1 & 0\\8 & -4\\1 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\\8x-4y\\x+2y\end{bmatrix}$ (UNIQUE CHOICE)