# Lecture 20: §3.7 Linear Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ II

GOAL: Study functions between (subspaces of) $\mathbb{R}^n$ & $\mathbb{R}^m$ that respect the vector space structure on both sides (addition of vectors & scalar multiplication)

Def: A function $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation if

(T1) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ in $\mathbb{R}^m$ for all $\vec{u}, \vec{v}$ in $\mathbb{R}^n$

& (T2) $T(\alpha \vec{u}) = \alpha T(\vec{u})$ in $\mathbb{R}^m$ for all $\vec{u}$ in $\mathbb{R}^n$, $\alpha$ in $\mathbb{R}$

Summary of results:

① All linear maps $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ are determined by multiplication by an $m \times n$ matrix $A$ ($f(\vec{v}) = A\vec{v}$).

Moreover $A = [\, f(\vec{e}_1) \; \cdots \; f(\vec{e}_n) \,]$. (columns are the image of the standard basis for $\mathbb{R}^n$ under $T$)

② Image of the map $f$ = Range of $A$ (Column Space)

③ Vectors $\vec{v}$ with $f(\vec{v}) = \vec{0}$ = Null Space of $A$.

**Exercise** Find a linear transformation $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ with

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\\4\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\1\\3\end{bmatrix} \;\&\; T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\2\\7\end{bmatrix}$$

# Null Space & Range

Each $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ linear transf has 2 subspaces associated to it

**Subspace 1** The Null-Space (or Kernel) of $T$ is the set of vectors $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ in $\mathbb{R}^n$ with $T\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ in $\mathbb{R}^m$ :

$$\mathcal{N}(T) = \{ \vec{x} \text{ in } \mathbb{R}^n : T(\vec{x}) = \vec{0} \}$$

**Example**: $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

**Subspace 2** The Range or image of $T$ is

$$R(T) = \{ \vec{y} \text{ in } \mathbb{R}^m : T(\vec{x}) = \vec{y} \text{ for some } \vec{x} \text{ in } \mathbb{R}^n \}$$

**Example**: $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Define: nullity of $T$ = $\dim \mathcal{N}(T)$

rank of $T$ = $\dim \mathcal{R}(T)$

Rank-nullity Theorem for $T$: nullity of $T$ + rank of $T = n$

$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

Example $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ $\quad T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 8 & -4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

# Matrix Representation

Theorem: Any linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ can be written as
$T(\vec{x}) = A\vec{x}$ for an $m \times n$ matrix $A = [T(\vec{e}_1) \ T(\vec{e}_2) \ \cdots T(\vec{e}_n)]$

Conclusion: $T$ is completely determined by its value on $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\}$
Moreover, we can prescribe any vector in $\mathbb{R}^m$ to be $T(\vec{e}_1), \cdots, T(\vec{e}_n)$

Q: What can we say if we pick a different basis for $\mathbb{R}^n$?

<u>Theorem</u>: Given a basis $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$ & ANY list of $n$ vectors $\{\vec{w}_1, \ldots, \vec{w}_n\}$ in $\mathbb{R}^m$, there is a unique linear transf. $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ with $T(\vec{v}_1) = \vec{w}_1, \ldots, T(\vec{v}_n) = \vec{w}_n$.

⚠ The result is false if we prescribe values on a non-basis $\{\vec{v}_1, \ldots \vec{v}_k\}$ of $\mathbb{R}^n$:

Observation: if $B = \{e_1, \dots, e_n\}$, then $T(\vec{x}) = [T(\vec{e_1}) \cdots T(\vec{e_n})] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

Here $[\vec{x}]_B = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

For a different basis $B = \{\vec{v_1}, \dots, \vec{v_n}\}$ $T(\vec{x}) = [T(\vec{v_1}) \cdots T(\vec{v_n})] [\vec{x}]_B$

Special example: "Taking coordinates" with respect to a basis $B$ is a linear transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $B = \{\vec{v_1}, \dots, \vec{v_n}\}$

$$\vec{v} \longrightarrow [\vec{v}]_B$$

$$[\vec{v} + \vec{w}]_B = [\vec{v}]_B + [\vec{w}]_B$$

$$[\alpha \vec{v}]_B = \alpha [\vec{v}]_B$$

Q: Why are matrix representations useful?

A: They allow for fast compositions.

**Theorem** Given $F: \mathbb{R}^n \longrightarrow \mathbb{R}^m$
& $G: \mathbb{R}^m \longrightarrow \mathbb{R}^s$ linear transformation, the

composition $T = G \circ F : \mathbb{R}^n \longrightarrow \mathbb{R}^s$ is also linear

$$\vec{x} \longmapsto G(F(\vec{x}))$$

Furthermore, if $F(\vec{v}) = A\vec{v}$ for $\vec{v}$ in $\mathbb{R}^n$
$G(\vec{w}) = B\vec{w}$ for $\vec{w}$ in $\mathbb{R}^m$, then the

matrix representing $G \circ T$ is $BA$ (size $s \times n$)

Very important: we multiply $B$ & $A$ in the same order as we compose $G$ & $F$.

**Thm** If $F(\vec{x}) = \underset{m \times n}{A}\,\vec{x}$ & $G(\vec{y}) = \underset{s \times n}{B}\,\vec{y}$ $\Rightarrow$ $G \circ F(\vec{x})$ linear

$$G \circ F(\vec{x}) = \underset{s \times m}{(BA)}\,\vec{x}$$