Lecture 20: §3.7 Linear Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ II

GOAL: Study functions between (subspaces of) $\mathbb{R}^n$ & $\mathbb{R}^m$ that respect the vector space structure on both sides (addition of vectors & scalar multiplication)

Def: A function $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation if

(T1) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ in $\mathbb{R}^m$ for all $\vec{u}, \vec{v}$ in $\mathbb{R}^n$

& (T2) $T(\alpha \vec{u}) = \alpha T(\vec{u})$ in $\mathbb{R}^m$ for all $\vec{u}$ in $\mathbb{R}^n$, α in $\mathbb{R}$

Summary of results:

① All linear maps $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ are determined by multiplication by an $m \times n$ matrix A ($f(\vec{v}) = A\vec{v}$).

Moreover $A = [\, f(\vec{e}_1) \cdots f(\vec{e}_n)\,]$. (columns are the image of the standard basis for $\mathbb{R}^n$ under T)

② Image of the map f = Range of A (Column Space)

③ Vectors $\vec{v}$ with $f(\vec{v}) = \vec{0}$ = Null Space of A.

Exercise Find a linear transformation $T:\mathbb{R}^2 \longrightarrow \mathbb{R}^3$ with

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\\4\end{bmatrix},\ T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}1\\1\\3\end{bmatrix}\ \&\ T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}2\\2\\7\end{bmatrix}$$

Solution: The first 2 conditions determine T uniquely:

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\left[\,T(\vec{e}_1)\ \ T(\vec{e}_2)\,\right]\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1&1\\0&1\\4&3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x+y\\y\\4x+3y\end{bmatrix}$$

To finish, we need to check the value of $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)$

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}1+1\\1\\4+3\end{bmatrix}=\begin{bmatrix}2\\1\\7\end{bmatrix}\neq\begin{bmatrix}2\\2\\7\end{bmatrix},$$ so there is no linear transformation $T:\mathbb{R}^2\longrightarrow\mathbb{R}^3$ satisfying the 3 prescribed conditions

Obs: The issue arises because $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right\}$ is l.d

so the relation $\vec{e}_1+\vec{e}_2-\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$ must be preserved under T:

$$T(\vec{e}_1)+T(\vec{e}_2)-T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}0\\0\end{bmatrix}$$

BUT $$\begin{bmatrix}1\\0\\4\end{bmatrix}+\begin{bmatrix}1\\1\\3\end{bmatrix}-\begin{bmatrix}2\\2\\7\end{bmatrix}=\begin{bmatrix}0\\-1\\0\end{bmatrix}\neq\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Null Space & Range

Each $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ linear transf has 2 subspaces associated to it

Subspace ① The Null-Space (or Kernel) of T is the set of vectors $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ in $\mathbb{R}^n$ with $T\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ in $\mathbb{R}^m$:

$$\mathcal{N}(T) = \{ \vec{x} \text{ in } \mathbb{R}^n : T(\vec{x}) = \vec{0} \} = \mathcal{N}(A) \text{ if } T(\vec{x}) = A\vec{x}$$

dimension = nullity of A

Example: $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \xrightarrow[R_2 \to R_2 - 4R_1]{} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2/-3 \\ R_1 - 2R_2}]{} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$ REF

x_1, x_2 dep, x_3 indep $\quad$ $x_1 = x_3$, $x_2 = -2x_3$ $\rightsquigarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \rightsquigarrow \mathcal{N}(T) = Sp\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right)$.

Subspace ② The Range or image of T is

$$R(T) = \{ \vec{y} \text{ in } \mathbb{R}^m : T(\vec{x}) = \vec{y} \text{ for some } \vec{x} \text{ in } \mathbb{R}^n \} = R(A)$$

dimension = rank of A

Example: $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ in $R(A)$

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \overset{G\text{-}J}{\sim} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$ REF $\quad$ x_1, x_2 dep so Basis for $R(A) = \left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$ dim 2 in $\mathbb{R}^2$ forces $R(A) = \mathbb{R}^2$.

Define: nullity of T = $\dim \mathcal{N}(T)$ (= nullity of A)

rank of T = $\dim \mathcal{R}(T)$ (= rank of A)

Rank-nullity Theorem for T: nullity of T + rank of T = n (= # cols A)

$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

Example $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ $\quad T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 8 & -4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

- $\mathcal{N}(T)$: $\begin{bmatrix} 1 & 0 \\ 8 & -4 \\ 1 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 8R_1} \begin{bmatrix} 1 & 0 \\ 0 & -4 \\ 0 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_2]{R_2 \to R_2/_{-4}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ REF

  so $x_1 = 0$, $x_2 = 0$ $\quad \mathcal{N}(T) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

  nullity $T = 0$

- $\mathcal{R}(T) = Sp\left(\begin{bmatrix} 1 \\ 8 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 2 \end{bmatrix} \right)$ it's a basis since $\begin{bmatrix} 1 & 0 \\ 8 & -4 \\ 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ x_1, x_2 dep vars
  - rank T = 2 − nullity T = 2 − 0 = 2
  - So it's a plane in $\mathbb{R}^3$. Equation? $\quad n = \begin{bmatrix} 1 \\ 8 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ -4 \\ 2 \end{bmatrix} = \begin{vmatrix} i & j & k \\ 1 & 8 & 1 \\ 0 & -4 & 2 \end{vmatrix} = \begin{bmatrix} 20 \\ -2 \\ -4 \end{bmatrix} = 2 \begin{bmatrix} 10 \\ -1 \\ -2 \end{bmatrix}$

  ⇝ Eqn: $\boxed{10x_1 - x_2 - 2x_3 = 0}$

$$\mathcal{R}(T) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : 10x_1 - x_2 - 2x_3 = 0 \right\}$$

Alternative: $\left[\begin{array}{cc|c} 1 & 0 & y_1 \\ 8 & -4 & y_2 \\ 1 & 2 & y_3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & y_1 \\ 0 & -4 & y_2 - 8y_1 \\ 0 & 2 & y_3 - y_1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & y_1 \\ 0 & 1 & y_2/4 - 2y_1 \\ 0 & 0 & \boxed{y_3 - y_1 + 2\left(\frac{y_2}{4} - 2y_1\right)} \end{array}\right]$ must be 0

⇝ $y_3 - 5y_1 + y_2/2 = 0$ or $\boxed{2y_3 - 10y_1 + y_2 = 0}$

Matrix Representation

<u>Theorem:</u> Any linear transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ can be written as
$T(\vec{x}) = A\vec{x}$ for an $m\times n$ matrix $A = [T(\vec{e_1})\ \ T(\vec{e_2})\ \cdots T(\vec{e_n})]$

<u>Example:</u> $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ $\quad T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_3 \\ 2x_2 - 5x_3 \end{bmatrix} \rightsquigarrow A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & -5 \end{bmatrix}$

Check $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1+0 \\ 0-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0+0 \\ 2-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0+2 \\ 0-5 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

<u>Conclusion:</u> T is completely determined by its values on $\{\vec{e_1}, \vec{e_2}, \ldots, \vec{e_n}\}$
Moreover, we can prescribe <u>any</u> vector in $\mathbb{R}^m$ to be $T(\vec{e_1}), \ldots, T(\vec{e_n})$

Q: What can we say if we pick a different basis for $\mathbb{R}^n$? A: Same thing!

<u>Theorem:</u> Given a basis $B = \{\vec{v_1}, \ldots, \vec{v_n}\}$ for $\mathbb{R}^n$ & ANY list of n vectors $\{\vec{w_1}, \ldots, \vec{w_n}\}$ in $\mathbb{R}^m$, there is a unique linear transf. $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ with $T(\vec{v_1}) = \vec{w_1}, \ldots, T(\vec{v_n}) = \vec{w_n}$.

<u>Example:</u> $T\Big(\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{v_1}}\Big) = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$ & $T\Big(\underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\vec{v_2}}\Big) = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$ uniquely determine T. $\left(T(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 8 & -4 \\ 1 & 2 \end{bmatrix} \vec{x}\right)$

because $\{\vec{v_1}, \vec{v_2}\}$ is a basis for $\mathbb{R}^2$.

Theorem: Given a basis $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ for $\mathbb{R}^n$ & ANY list of n vectors $\{\vec{w}_1, \dots, \vec{w}_n\}$ in $\mathbb{R}^m$, there is a unique linear transf. $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ with $T(\vec{v}_1) = \vec{w}_1, \dots, T(\vec{v}_n) = \vec{w}_n$.

Proof: Every $\vec{x}$ in $\mathbb{R}^n$ is a UNIQUE linear combination of $B = \{\vec{v}_1, \dots \vec{v}_n\}$

$$\vec{x} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_n \vec{v}_n$$

Write $[\vec{x}]_B = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$

Apply T:

$$T(\vec{x}) = \alpha_1 T(\vec{v}_1) + \alpha_2 T(\vec{v}_2) + \dots + \alpha_n T(\vec{v}_n)$$
$$= \alpha_1 \vec{w}_1 + \alpha_2 \vec{w}_2 + \dots + \alpha_n \vec{w}_n.$$

$\rightsquigarrow T(\vec{x}) = \underbrace{[\vec{w}_1 \dots \vec{w}_n]}_{\text{matrix of } T(\vec{v}_1) \dots T(\vec{v}_n)} [\vec{x}]_B$

- T is linear

$$\begin{aligned} & \vec{x} = \alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n \\ + \ & \vec{y} = \beta_1 \vec{v}_1 + \dots + \beta_n \vec{v}_n \\ \hline & \vec{x}+\vec{y} = (\alpha_1+\beta_1)\vec{v}_1 + \dots + (\alpha_n+\beta_n)\vec{v}_n \end{aligned}$$

$\rightsquigarrow T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$

- Similarly $T(a\vec{x}) = a T(\vec{x})$ for all scalars a $\square$

⚠ The result is false if we prescribe values on a non-basis $\{\vec{v}_1, \dots \vec{v}_k\}$ of $\mathbb{R}^n$:

① If $\{\vec{v}_1, \dots, \vec{v}_k\}$ are l.d., then $\{\vec{w}_1, \dots, \vec{w}_k\}$ must satisfy the same dependencies (Example, on slide 2)

② If $\{\vec{v}_1, \dots \vec{v}_k\}$ are l.i. but $k < n$, they don't span, so T exists, but it's not unique.

Eg: $T: \mathbb{R}^2 \longrightarrow \mathbb{R}$ with $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = 1$. $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$ can be whatever we want!

Observation: if $B=\{e_1 \dots e_n\}$, then $T(\vec{x}) = \left[T(\vec{e_1}) \cdots T(\vec{e_n})\right]\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}$

Here $[\vec{X}]_B = \begin{bmatrix}x_1\\ \vdots \\ x_n\end{bmatrix}$

For a different basis $B=\{\vec{v}_1, \dots, \vec{v}_n\}$ $\quad T(\vec{x}) = \left[T(\vec{v}_1) \cdots T(\vec{v}_n)\right][\vec{x}]_B$

Special example: "Taking coordinates" with respect to a basis B is a linear transformation

$B = \{\vec{v}_1, \dots, \vec{v}_n\}$

$$T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$\vec{v} \longrightarrow [\vec{v}]_B$$

$[\vec{v}+\vec{w}]_B = [\vec{v}]_B + [\vec{w}]_B$

$[\alpha \vec{v}]_B = \alpha [\vec{v}]_B$

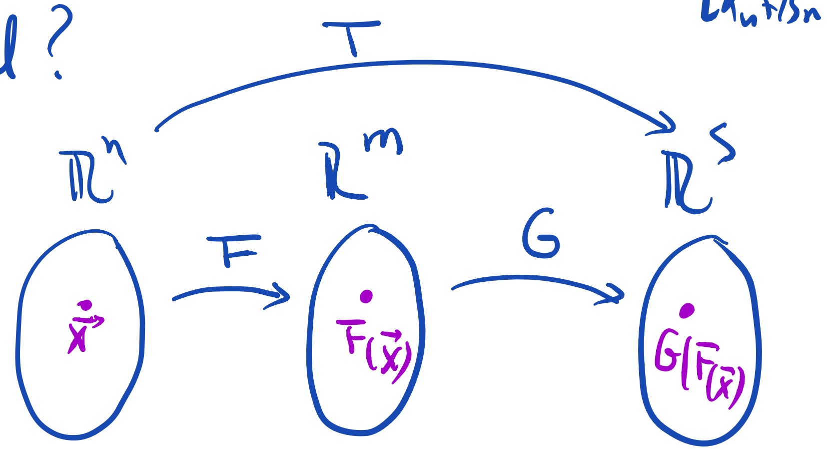
$\vec{v} = \alpha_1 \vec{v}_1 + \cdots + \alpha_n \vec{v}_n \qquad [\vec{v}]_B = \begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{bmatrix}$

$+\ \vec{w} = \beta_1 \vec{v}_1 + \cdots + \beta_n \vec{v}_n \qquad [\vec{w}]_B = \begin{bmatrix}\beta_1\\ \vdots \\ \beta_n\end{bmatrix}$

$\vec{v}+\vec{w} = (\alpha_1+\beta_1)\vec{v}_1 + \cdots + (\alpha_n+\beta_n)\vec{v}_n \qquad [v+\vec{w}]_B = \begin{bmatrix}\alpha_1+\beta_1\\ \vdots \\ \alpha_n+\beta_n\end{bmatrix}$

Q: Why are matrix representations useful?

A: They allow for fast compositions.



$$T(\vec{x}) = (G \circ F)(\vec{x}) = G\big(\underbrace{F(\vec{x})}_{\text{in } \mathbb{R}^m}\big)$$

Theorem Given $F: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ & $G: \mathbb{R}^m \longrightarrow \mathbb{R}^s$ linear transformation, the composition $T = G \circ F : \mathbb{R}^n \longrightarrow \mathbb{R}^s$ is also linear

$$\vec{x} \longmapsto G(F(\vec{x}))$$

Furthermore, if $F(\vec{v}) = A\vec{v}$ for $\vec{v}$ in $\mathbb{R}^n$, $G(\vec{w}) = B\vec{w}$ for $\vec{w}$ in $\mathbb{R}^m$, then the matrix representing $G \circ T$ is BA (size $s \times n$)

Very important: we multiply B & A in the same order as we compose G & F.

Ex: $F: \mathbb{R}^4 \longrightarrow \mathbb{R}^2$, $\quad F\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_3 + x_4 \end{bmatrix} \rightsquigarrow F(\vec{x}) = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \vec{x}$

$G: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $\quad G\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} y_1 \\ y_1 + 5y_2 \\ -y_1 + y_2 \end{bmatrix} \rightsquigarrow G(\vec{y}) = \begin{bmatrix} 1 & 0 \\ 1 & 5 \\ -1 & 1 \end{bmatrix} \vec{y}$

$$G \circ F\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = G\left(\begin{bmatrix} x_1 - x_2 \\ x_3 + x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ (x_1 - x_2) + 5(x_3 + x_4) \\ -(x_1 - x_2) + (x_3 + x_4) \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_1 - x_2 + 5x_3 + 5x_4 \\ -x_1 + x_2 + x_3 + x_4 \end{bmatrix}$$

$y_1 = x_1 - x_2$ & $y_2 = x_3 + x_4$

$$\rightsquigarrow G \circ F(\vec{x}) = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 5 & 5 \\ -1 & 1 & 1 & 1 \end{bmatrix} \vec{x} \quad \text{linear} \quad \& \quad \begin{bmatrix} 1 & 0 \\ 1 & 5 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 5 & 5 \\ -1 & 1 & 1 & 1 \end{bmatrix}$$

Thm If $F(\vec{x}) = \underset{m\times n}{A}\vec{x}$ & $G(\vec{y}) = \underset{s\times n}{B}\vec{y}$ ⇝ $G\circ F(\vec{x})$ linear

$G\circ F(\vec{x}) = \underset{s\times m}{(BA)}\vec{x}$

Why? We need to check the 2 properties:

- $G\circ F(\vec{u}+\vec{v}) = G(F(\vec{u}+\vec{v})) \underset{F \text{ linear}}{=} G(\underbrace{F(\vec{u})}_{=\vec{w}_1} + \underbrace{F(\vec{v})}_{\vec{w}_2}) \underset{G \text{ linear}}{=}$

$= G(F(\vec{u})) + G(F(\vec{v})) = G\circ F(\vec{u}) + G\circ F(\vec{v})$

- $G\circ F(\alpha\vec{u}) = G(F(\alpha\vec{u})) \underset{F \text{ linear}}{=} G(\alpha \underbrace{F(\vec{u})}_{=\vec{v}}) \underset{G \text{ linear}}{=} \alpha G(\vec{v}) = \alpha\, G\circ F(\vec{u})$

To finish: matrix representation for $G\circ F$ is $[G\circ F(\vec{e}_1) \cdots G\circ F(\vec{e}_n)]$

But $G\circ F(\vec{e}_1) = G(F(\vec{e}_1)) = G\left(A\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix}\right) = G(\text{col}_1(A)) = B\,\text{col}_1(A)$

$G\circ F(\vec{e}_2) = G(F(\vec{e}_2)) = G\left(A\begin{bmatrix}0\\1\\0\\ \vdots\\0\end{bmatrix}\right) = G(\text{col}_2(A)) = B\,\text{col}_2(A)$

$\vdots$

$G\circ F(\vec{e}_n) = G(F(\vec{e}_n)) = G\left(A\begin{bmatrix}0\\ \vdots\\0\\1\end{bmatrix}\right) = G(\text{col}_n(A)) = B\,\text{col}_n(A)$

So $[G\circ F(\vec{e}_1) \cdots G\circ F(e_n)] = [\underbrace{B\,\text{col}_1 A}_{=\text{col}_1(BA)} \cdots \underbrace{B\,\text{col}_n A}_{=\text{col}_n(BA)}] = BA$

by def of matrix mult. BA has the correct size.