

Lecture 21: §5.1-5.2 (Abstract) Vector Spaces

Motivation: So far, we have constructed $\mathbb{R}^n$ & subspaces $\mathcal{V}$ of $\mathbb{R}^n$

① $\mathcal{V} = \mathbb{R}^n$

⑤ $\mathcal{V} = \mathcal{R}(A) = \text{Sp}(\text{Col}_1 A, \dots, \text{Col}_n A) \text{ in } \mathbb{R}^m$
 $\text{Range / Col Space of } A$
 $= \mathcal{R}(T) \text{ where } T: \mathbb{R}^n \rightarrow \mathbb{R}^m, T(\vec{v}) = A\vec{v}$

② $\mathcal{V} = \{\vec{0}\}$

③ $\mathcal{V} = \text{Sp}(\vec{v}_1, \dots, \vec{v}_r)$

⑥ $\mathcal{V} = \mathcal{N}(A) = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{0} \text{ in } \mathbb{R}^m \right\} = \mathcal{N}(T)$
 $\text{Nullspace of } A$

④ $\mathcal{V} = \text{Row Space}(A) = \text{Sp}(\text{Row}_1 A, \text{Row}_2 A, \dots, \text{Row}_m A) = \mathcal{R}(A^T)$
 $\text{Row Space of } A$

Common feature: (1) $\vec{0}$ in $\mathcal{V}$

(2) We have addition $+$ of vectors & if $\vec{u}$ & $\vec{v}$ are in $\mathcal{V}$,
so is $\vec{u} + \vec{v}$

(3) We can scalar multiply any $\vec{u}$ in $\mathcal{V}$ & remain in $\mathcal{V}$

(4) $+$ & scalar mult interact nicely (we have nice algebraic properties, inherited from $\mathbb{R}^n$)

These properties appear in many other contexts in Mathematics.

Example ①: A set of solutions to a homogeneous differential equation.

(*) $y''_{(x)} - y_{(x)} = 0$ has 2 types of solutions

Solution 1: $y_1(x) = e^x$

Solution 2: $y_2(x) = -e^{-x}$

$\Rightarrow$ Any linear combination of these 2 solns is also a solution!

$$y(x) = ae^x + b(-e^{-x}) \quad \text{with } a, b \in \mathbb{R}$$

$$y'(x) = ae^x + b(-1)(-1)e^{-x} = ae^x + be^{-x}$$

$$y''(x) = ae^x - be^{-x} = y(x) \quad \checkmark$$

$\Rightarrow$ solutions 1 & 2 are "linearly independent"

$$0 = ae^x + b(-e^{-x}) \quad \text{for all } x \in \mathbb{R}$$

Evaluate at special x 's to confirm $a=b=0$

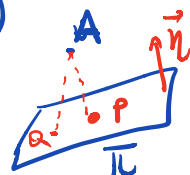
• At $x=0$: $0 = a - b \Rightarrow a=b$

• At $x=1$: $0 = a \cdot e - b \cdot e^{-1} = ae - \frac{b}{e} = \frac{ae^2 - b}{e} = a \frac{e^2 - 1}{e} \stackrel{x=0}{\text{so } a=0}$

• We can show solutions to (*) = $\{y_{(x)} : y'' + y = 0\} = \text{Sp}(e^x, -e^{-x})$.

Example ②: Fix $A \in \mathbb{R}^3$ and a plane π through $(0,0,0)$

Q: What is the closest point p to A in the plane?



A: Need to minimize the distance $d(A, P) = \| \vec{AP} \| \quad P \in \pi$
 $= \sqrt{\langle \vec{AP}, \vec{AP} \rangle}$

Geometry says $\vec{AP} \perp$ plane, so it must be proportional to the normal $\vec{n}$.

We have other situations where we want to minimize distances:

• Fix the space $\mathcal{V}$ of all continuous functions on $[0,1]$

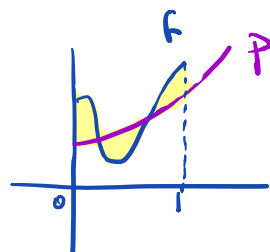
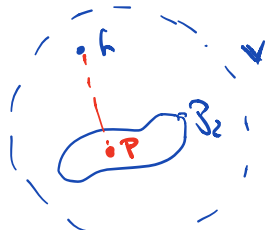
We want to approximate a given $f: [0,1] \rightarrow \mathbb{R}$ by a polynomial

$\mathcal{P}_2$ of degree ≤ 2 , minimizing the "gap" between P & f

$$\mathcal{P}_2 = \{ a + bx + cx^2 : a, b, c \in \mathbb{R} \}$$

$$Error = \| f - P \| = \int_0^1 (f(x) - P(x))^2 dx$$

$$\langle g, h \rangle = \int_0^1 g(x) h(x) dx \quad \text{is an inner product in } \mathcal{V}.$$



"area" of shaded part.
measures the gap.

Vector Spaces

Q: What defines an abstract vector space?

A. We need a set $\mathbb{V}$. We call each element of $\mathbb{V}$ a vector $\vec{v}$.

We need to have two operations on $\mathbb{V}$:

① Addition: $+$: $\mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{V}$
 $(\vec{u}, \vec{v}) \longmapsto \vec{u} + \vec{v}$ (add two vectors to get a new vector)

② Scalar Multiplication: $\cdot$: $\mathbb{R} \times \mathbb{V} \longrightarrow \mathbb{V}$
 $(\alpha, \vec{v}) \longrightarrow \alpha \vec{v}$
scalar $\nearrow$ vector $\nwarrow$ $\alpha \vec{v}$ is a new vector

• We need $+$ & $\cdot$ to have nice properties

Examples:

- ① $\mathbb{R}^n$ & subspaces $\mathbb{V}$ of $\mathbb{R}^n$ (usual $+$ & $\cdot$)
- ② $\text{Mat}_{n \times m}(\mathbb{R})$ = $n \times m$ matrices with real entries (—)
- ③ Polynomials with real coefficients eg $\mathcal{P}_2 = \{ax^2 + bx + c : a, b, c\}$

Defining properties for vector spaces

Def A set V with addition $+$ & scalar mult. is a vector space if it satisfies

① Closure Properties: (C1) $\vec{x}, \vec{y}$ in V , then $\vec{x} + \vec{y}$ in V

(C2) $\vec{x}$ in V , then $\alpha \vec{x}$ in V

② Addition Properties: (A1) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ (Commutative)

(A2) $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$ (Associative)

(Neutral element) $\leftarrow$ (A3) $\vec{0}$ in V satisfies $\vec{x} + \vec{0} = \vec{0} + \vec{x} = \vec{x}$ for all $\vec{x}$.

(Additive Inverses) $\leftarrow$ (A4) Given $\vec{x}$ in V we can find " $-\vec{x}$ " in V with $\vec{x} + (-\vec{x}) = \vec{0}$ (here " $-\vec{x}$ " = $(-1)\vec{x}$)

③ Scalar Mult. Properties: (M1) $\alpha(\beta \vec{x}) = (\alpha\beta) \vec{x}$ (Associative)

(M2) $\alpha(\vec{x} + \vec{y}) = \alpha \vec{x} + \alpha \vec{y}$ (Distributive 1)

(M3) $(\alpha + \beta) \vec{x} = \alpha \vec{x} + \beta \vec{x}$ (Distributive 2)

(M4) follows from (C2)

(M4) $1 \vec{x} = \vec{x}$ for all $\vec{x}$

Example 1: $\text{Mat}_{m \times n}(\mathbb{R})$

- + A, B of size $m \times n \rightsquigarrow A+B$ of size $m \times n$ $(A+B)_{ij} = A_{ij} + B_{ij}$
(+ entry-by-entry)
- $\cdot$ A of size $m \times n$, α scalar $\rightsquigarrow \alpha A$ of size $m \times n$ & $(\alpha A)_{ij} = \alpha A_{ij}$
($\cdot$ entry-by-entry)

$\mathbf{0} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & & \\ 0 & & 0 \end{bmatrix}$ zero matrix, $(-A) = (-1) \cdot A$.

In terms of operations, we can think of $\text{Mat}_{m \times n}(\mathbb{R})$ as $\mathbb{R}^{m \cdot n}$
(write all entries as 1 long column (write col 1, then col 2, ...))

Since $\mathbb{R}^{m \cdot n}$ is a vector space, so is $\text{Mat}_{m \times n}(\mathbb{R})$. [the operations are done entry-by-entry so how we "arrange" the entries is irrelevant]

Non-example $\mathbb{V} = \{A \text{ } 2 \times 2 \text{ \& singular}\}$ (C1) fails

Why? $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ & $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ in $\mathbb{V}$ but $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is not in $\mathbb{V}$.

Example 2: Polynomials

Set $\mathcal{P}_n = \{ \text{polynomials of degree at most } n = \{ a_0 + a_1 x + \dots + a_n x^n : a_0, a_1, \dots, a_n \text{ in } \mathbb{R} \} \}$

± Coefficient-by-coefficient

$$\begin{aligned} P(x) &= a_0 + a_1 x + \dots + a_n x^n \\ + Q(x) &= b_0 + b_1 x + \dots + b_n x^n \end{aligned}$$

$$P + Q(x) = (a_0 + b_0) + (a_1 + b_1)x + \dots + (a_n + b_n)x^n$$

≡ Coefficient-by-coefficient $\alpha P(x) = (\alpha a_0) + (\alpha a_1)x + \dots + (\alpha a_n)x^n$

• $0 = 0 + 0 \cdot x + \dots + 0 x^{n-1} = \text{zero polynomial}$

• We identify P in $\mathcal{P}_n$ with its list of coefficients $\begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix}$ in $\mathbb{R}^{n+1}$

So $\mathcal{P}_n$ behaves like $\mathbb{R}^{n+1}$ for all practical purposes, so the 10 properties are satisfied!

Ex $\mathcal{P}_2 = \{ a_0 + a_1 x + a_2 x^2 \} = \text{Sp}(1, x, x^2)$

Nm-example $\mathbb{W} = \{ P \in \mathcal{P}_2 \text{ with } P(0) = 1 \}$ (so $P = 1 + a_1 x + a_2 x^2$)
 (A3) fails $\mathbb{0}$ is NOT in $\mathbb{W}$.

Example $V = \mathbb{R}^2$ with usual scalar mult, but funky $+$:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 - 1 \\ u_2 + v_2 - 1 \end{bmatrix}$$

• (C1) & (C2) 2 Closure properties hold. (We get vectors in $\mathbb{R}^2$)

• (A1) $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$ holds

• (A2) $\vec{u} \oplus (\vec{v} \oplus \vec{w}) = \left[\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \oplus \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right) \right] = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \begin{bmatrix} v_1 + w_1 - 1 \\ v_2 + w_2 - 1 \end{bmatrix}$

$= \begin{bmatrix} u_1 + (v_1 + w_1 - 1) - 1 \\ u_2 + (v_2 + w_2 - 1) - 1 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 + w_1 - 2 \\ u_2 + v_2 + w_2 - 2 \end{bmatrix}$

same $\parallel$

$$(\vec{u} \oplus \vec{v}) \oplus \vec{w} = \begin{bmatrix} u_1 + v_1 - 1 \\ u_2 + v_2 - 1 \end{bmatrix} \oplus \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} (u_1 + v_1 - 1) + w_1 - 1 \\ (u_2 + v_2 - 1) + w_2 - 1 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 + w_1 - 2 \\ u_2 + v_2 + w_2 - 2 \end{bmatrix}$$

• (A3) Neutral element $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} u_1 + a - 1 \\ u_2 + b - 1 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \leadsto a = b = 1$

So $\mathbf{0}_{\text{new}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

• (A4) Additive inverse $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for $\begin{cases} u_1 + a - 1 = 1 \\ u_2 + b - 1 = 1 \end{cases}$

So $\text{Inverse of } \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -u_1 + 2 \\ -u_2 + 2 \end{bmatrix}$

$V = \mathbb{R}^2$ with usual scalar mult but $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 - 1 \\ u_2 + v_2 - 1 \end{bmatrix}$

(11) Associative $\alpha(\beta \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}) = (\alpha\beta) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ ✓ inherited from $\mathbb{R}^2$ (same scalar mult!)

(14) $1 \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ also inherited from $\mathbb{R}^2$ ✓

(12) Distributive 1 $\alpha(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}) = \alpha \begin{bmatrix} u_1 + v_1 - 1 \\ u_2 + v_2 - 1 \end{bmatrix} = \begin{bmatrix} \alpha u_1 + \alpha v_1 - \alpha \\ \alpha u_2 + \alpha v_2 - \alpha \end{bmatrix}$

$$\alpha \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \alpha \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \alpha u_1 \\ \alpha u_2 \end{bmatrix} \oplus \begin{bmatrix} \alpha v_1 \\ \alpha v_2 \end{bmatrix} = \begin{bmatrix} \alpha u_1 + \alpha v_1 - 1 \\ \alpha u_2 + \alpha v_2 - 1 \end{bmatrix}$$

So the property fails when $\alpha \neq 1$.

(13) Distributive 2 also fails

$$(\alpha + \beta) \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} (\alpha + \beta) u_1 \\ (\alpha + \beta) u_2 \end{bmatrix} = \begin{bmatrix} \alpha u_1 + \beta u_1 \\ \alpha u_2 + \beta u_2 \end{bmatrix}$$

$$\alpha \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} \oplus \beta \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \alpha u_1 \\ \alpha u_2 \end{bmatrix} \oplus \begin{bmatrix} \beta u_1 \\ \beta u_2 \end{bmatrix} = \begin{bmatrix} \alpha u_1 + \beta u_1 - 1 \\ \alpha u_2 + \beta u_2 - 1 \end{bmatrix}$$

Conclusion: $\mathbb{R}^2$ with $\oplus$ & usual $\cdot$ is NOT a vector space.