Lecture 21: §§5.1–5.2 (Abstract) Vector Spaces

Motivation: So far, we have constructed $\mathbb{R}^N$ & subspaces $\mathbb{V}$ of $\mathbb{R}^N$

① $\mathbb{V} = \mathbb{R}^n$

② $\mathbb{V} = \{\vec{0}\}$

③ $\mathbb{V} = Sp(\vec{v}_1, \ldots, \vec{v}_r)$

④ $\mathbb{V} = \text{Row Space}\underset{m\times n}{(A)} = Sp(\text{Row}_1 A, \text{Row}_2 A, \ldots \text{Row}_m A)$

⑤ $\mathbb{V} = \mathcal{R}\underset{m\times n}{(A)} = Sp(\text{Col}_1 A, \ldots, \text{Col}_n A)$ in $\mathbb{R}^m$

Range / Col Space of A

$= \mathcal{R}(T)$ where $T: \mathbb{R}^n \to \mathbb{R}^m$ $T(\vec{v}) = A\vec{v}$

⑥ $\mathbb{V} = \mathcal{N}\underset{m\times n}{(A)} = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{0} \text{ in } \mathbb{R}^m \right\} = \mathcal{N}(T)$

Nullspace of A

Common feature: (1) $\vec{0}$ in $\mathbb{V}$

(2) We have addition + of vectors & if $\vec{u}$ & $\vec{v}$ are in $\mathbb{V}$, so is $\vec{u} + \vec{v}$

(3) We can scalar multiply any $\vec{u}$ in $\mathbb{V}$ & remain in $\mathbb{V}$

(4) + & scalar mult interact nicely (we have nice algebraic properties, inherited from $\mathbb{R}^N$)

These properties appear in many other contexts in Mathematics.

Example 1: A set of solutions to a homogeneous differential equation.

(*) $\boxed{y''_{(x)} - y_{(x)} = 0}$ has 2 types of solutions

Solution 1: $y_1(x) = e^x$ Solution 2 $y_2(x) = -e^{-x}$.

$\rightsquigarrow$ Any linear combination of these 2 solns is also a solution!

$$\boxed{y(x) = ae^x + b(-e^{-x})} \quad \text{with } a, b \text{ in } \mathbb{R}$$

$$y'(x) = ae^x + b(-1)(-1)e^{-x} = ae^x + be^{-x}$$

$$y''(x) = ae^x - be^{-x} = y(x) \checkmark$$

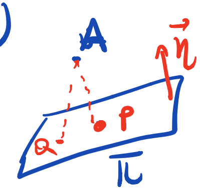
$\rightsquigarrow$ solutions 1 & 2 are "linearly independent"

$$0 = ae^x + b(-e^{-x}) \quad \text{for all } x \text{ in } \mathbb{R}$$

Evaluate at special x's to confirm $a = b = 0$

- At $x=0$: $0 = a - b \rightsquigarrow a = b$
- At $x=1$ $0 = a \cdot e - b \cdot e^{-1} = ae - \frac{b}{e} = \frac{ae^2 - b}{e} = a\underbrace{\frac{e^2-1}{e}}_{\neq 0}$ so $a=0$
- We can show solutions to (*) $= \{y_{(x)} : y'' + y = 0\} = Sp(e^x, -e^{-x})$.

Example ②: Fix A in $\mathbb{R}^3$ and a plane π through $(0,0,0)$



Q: What is the closest point P to A in the plane?

A: Need to minimize the distance $d(A,P) = \| \vec{AP} \|$ $\quad P$ in π

$$= \sqrt{\langle \vec{AP}, \vec{AP} \rangle}$$

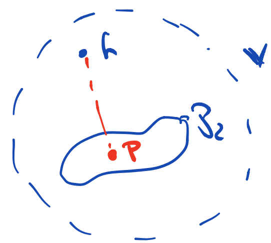
Geometry says $\vec{AP} \perp$ plane, so it must be proportional to the normal $\vec{n}$.

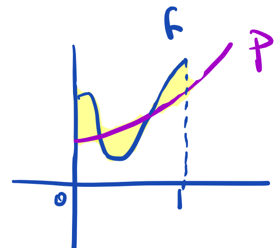
We have other situations where we want to minimize distances:

- Fix the space V of all continuous functions over $[0,1]$

We want to approximate a given $f : [0,1] \longrightarrow \mathbb{R}$ by a polynomial $P_{(x)}$ of degree ≤ 2, minimizing the "gap" between P & f

$$\mathcal{P}_2 = \{ a + bx + cx^2 \; : a, b, c \text{ in } \mathbb{R} \}$$





$$\text{Error} = \| f - P \| = \int_0^1 (f_{(x)} - P_{(x)})^2 \, dx$$

"area" of shaded part measures the gap.

$$\langle g, h \rangle = \int_0^1 g_{(x)} h_{(x)} \, dx$$ is an inner product on V.

$\boxed{\text{Vector Spaces}}$

Q: What defines an abstract vector space?

A. We need a set $\boxed{V}$. We call each element of V a vector $\vec{v}$

. We need to have two operations on V:

1. Addition: $+ : V \times V \longrightarrow V$ (add two vectors to get a new vector)

$$(\vec{u}, \vec{v}) \longmapsto \vec{u} + \vec{v}$$

2. Scalar Multiplication: $\cdot\ \mathbb{R} \times V \longrightarrow V$ ($\alpha\vec{v}$ is a new vector)

$$(\underbrace{\alpha}_{\text{scalar}}, \underbrace{\vec{v}}_{\text{vector}}) \longrightarrow \alpha\vec{v}$$

. We need $+$ & $\cdot$ to have nice properties

Examples:
1. $\mathbb{R}^n$ & subspaces V of $\mathbb{R}^n$ (usual $+$ & $\cdot$)
2. $\text{Mat}_{n\times m}(\mathbb{R})$ = $n\times m$ matrices with real entries (——)
3. Polynomials with real coefficients eg $\mathcal{P}_2 = \{ax^2+bx+c : a,b,c\}$

Defining properties for vector spaces

Def A set $\mathbb{V}$ with addition $+$ & scalar mult. is a vector space if it satisfies

① **Closure Properties:**

(C1) $\vec{x}, \vec{y}$ in $\mathbb{V}$, then $\vec{x}+\vec{y}$ in $\mathbb{V}$

(C2) $\vec{x}$ in $\mathbb{V}$, then $\alpha \cdot \vec{x}$ in $\mathbb{V}$

② **Addition Properties:**

(A1) $\vec{x}+\vec{y} = \vec{y}+\vec{x}$ (Commutative)

(A2) $\vec{x}+(\vec{y}+\vec{z}) = (\vec{x}+\vec{y})+\vec{z}$ (Associative)

(Neutral element) ⟵ (A3) $\vec{0}$ in $\mathbb{V}$ satisfies $\vec{x}+\vec{0} = \vec{0}+\vec{x} = \vec{x}$ for all $\vec{x}$.

(Additive Inverses) ⟵ (A4) Given $\vec{x}$ in $\mathbb{V}$ we can find "$-\vec{x}$" in $\mathbb{V}$ with $\vec{x}+(-\vec{x}) = \vec{0}$ (here "$-x$" $= (-1)\vec{x}$)

③ **Scalar Mult. Properties:**

(M1) $\alpha(\beta\vec{x}) = (\alpha\beta)\vec{x}$ (Associative)

(M2) $\alpha(\vec{x}+\vec{y}) = \alpha\vec{x}+\alpha\vec{y}$ (Distributive 1)

(M3) $(\alpha+\beta)\vec{x} = \alpha\vec{x}+\beta\vec{x}$ (—— 2)

(M4) $1\vec{x} = \vec{x}$ for all $\vec{x}$

(A4) follows from (C2)

Example 1: $\mathrm{Mat}_{m\times n}(\mathbb{R})$

$\underline{\underline{+}}$ A, B of size $m\times n$ $\rightsquigarrow$ $A+B$ of size $m\times n$ $\quad (A+B)_{ij} = A_{ij} + B_{ij}$

(+ entry-by-entry)

$\underline{\underline{\cdot}}$ A of size $m\times n$, α scalar $\rightsquigarrow$ αA of size $m\times n$ & $(\alpha A)_{ij} = \alpha A_{ij}$

(· entry-by-entry)

$\mathbb{0} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & & 0 \end{bmatrix}$ zero matrix, $\quad (-A) = (-1)\cdot A$.

In terms of operations, we can thing of $\mathrm{Mat}_{m\times n}(\mathbb{R})$ as $\mathbb{R}^{m\cdot n}$

(write all entries as 1 long column (write col 1, then col 2, ...))

Since $\mathbb{R}^{m\cdot n}$ is a vector space, so is $\mathrm{Mat}_{m\times n}(\mathbb{R})$. [the operations are done entry-by-entry so how we "arrange" the entries is irrelevant]

<u>Non-example</u> $\quad \mathbb{V} = \{A \;\; 2\times 2 \text{ \& singular}\}$ $\quad$ (C1) fails

Why? $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ & $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ in $\mathbb{V}$ <u>but</u> $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is <u>not</u> in $\mathbb{V}$.

Example 2: Polynomials

Set $\boxed{\mathcal{P}_n}$ = { polynomials of degree at most n } = $\{ a_0 + a_1 x + \dots + a_n x^{n-1} :$
$a_0, a_1, \dots, a_n$ in $\mathbb{R}\}$

$\pm$ Coefficient-by-coefficient

$$P(x) = a_0 + a_1 x + \dots + a_n x^n$$
$$+\quad Q(x) = b_0 + b_1 x + \dots + b_n x^n$$

$$\boxed{P+Q(x)} = (a_0+b_0) + (a_1+b_1)x + \dots + (a_n+b_n)x^n$$

$\doteq$ Coefficient-by-coefficient $\quad \alpha P(x) = (\alpha a_0) + (\alpha a_1)x + \dots + (\alpha a_n)x^n$

- $\mathbb{O} = 0 + 0 \cdot x + \dots + 0 x^{n-1}$ = zero polynomial
- We identify P in $\mathcal{P}_n$ will its list of coefficients $\begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix}$ in $\mathbb{R}^{n+1}$

So $\mathcal{P}_n$ behaves like $\mathbb{R}^{n+1}$ for all practical purposes, so the 10 properties are satisfied!

Ex $\mathcal{P}_2 = \{ a_0 + a_1 x + a_2 x^2 \} = Sp(1, x, x^2)$.

Non-example $\quad \mathbb{V} = \{ P$ in $\mathcal{P}_2$ with $P(0) = 1 \}$ $\quad (\text{so } P = 1 + a_1 x + a_2 x^2)$

(A3) fails $\qquad \mathbb{O}$ is NOT in $\mathbb{V}$.

Example $\boxed{\mathbb{V} = \mathbb{R}^2}$ with usual scalar mult, but funky $+$:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 - 1 \\ u_2 + v_2 - 1 \end{bmatrix}$$

- (C1) & (C2) 2 Closure properties hold. (We get vectors in $\mathbb{R}^2$)
- (A1) $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$ holds
- (A2) $\vec{u} \oplus (\vec{v} \oplus \vec{w}) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \oplus \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \begin{bmatrix} v_1 + w_1 - 1 \\ v_2 + w_2 - 1 \end{bmatrix}$

$$= \begin{bmatrix} u_1 + (v_1 + w_1 - 1) - 1 \\ u_2 + (v_2 + w_2 - 1) - 1 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 + w_1 - 2 \\ u_2 + v_2 + w_2 - 2 \end{bmatrix}$$

same $\Vert$

$$(\vec{u} \oplus \vec{v}) \oplus \vec{w} = \begin{bmatrix} u_1 + v_1 - 1 \\ u_2 + v_2 - 1 \end{bmatrix} \oplus \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} (u_1 + v_1 - 1) + w_1 - 1 \\ (u_2 + v_2 - 1) + w_2 - 1 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 + w_1 - 2 \\ u_2 + v_2 + w_2 - 2 \end{bmatrix}$$

- (A3) Neutral element $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} u_1 + a - 1 \\ u_2 + b - 1 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \rightsquigarrow a = b = 1$

So $\boxed{\mathbb{O}_{new} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

- (A4) Additive inverse $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ forces $\begin{cases} u_1 + a - 1 = 1 \\ u_2 + b - 1 = 1 \end{cases}$

So $\boxed{\text{Inverse of } \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -u_1 + 2 \\ -u_2 + 2 \end{bmatrix}}$

$$\boxed{\mathbb{V} = \mathbb{R}^2 \text{ with usual scalar mult but } \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 - 1 \\ u_2 + v_2 - 1 \end{bmatrix}}$$

(M1) <u>Associative</u> $\alpha\left(\beta \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = (\alpha\beta) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ ✓ inherited from $\mathbb{R}^2$ (same scalar mult!)

(M4) $1 \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ also inherited from $\mathbb{R}^2$ ✓

(M2) <u>Distributive 1</u> $\alpha\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \alpha \begin{bmatrix} u_1 + v_1 - 1 \\ u_2 + v_2 - 1 \end{bmatrix} = \begin{bmatrix} \alpha u_1 + \alpha v_1 - \alpha \\ \alpha u_2 + \alpha v_2 - \alpha \end{bmatrix}$

$$\alpha \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \oplus \alpha \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \alpha u_1 \\ \alpha u_2 \end{bmatrix} \oplus \begin{bmatrix} \alpha v_1 \\ \alpha v_2 \end{bmatrix} = \begin{bmatrix} \alpha u_1 + \alpha v_1 - 1 \\ \alpha u_2 + \alpha v_2 - 1 \end{bmatrix}$$

So the property fails when $\alpha \neq 1$.

(M3) <u>Distributive 2</u> also fails

$$(\alpha + \beta) \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} (\alpha+\beta) u_1 \\ (\alpha+\beta) u_2 \end{bmatrix} = \begin{bmatrix} \alpha u_1 + \beta u_1 \\ \alpha u_2 + \beta u_2 \end{bmatrix}$$

$$\alpha \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} \oplus \beta \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \alpha u_1 \\ \alpha u_2 \end{bmatrix} \oplus \begin{bmatrix} \beta u_1 \\ \beta u_2 \end{bmatrix} = \begin{bmatrix} \alpha u_1 + \beta u_1 - 1 \\ \alpha u_2 + \beta u_2 - 1 \end{bmatrix}$$

<u>Conclusion</u>: $\mathbb{R}^2$ with $\oplus$ & usual $\cdot$ is NOT a vector space.