Lecture 22: §5.3 Subspaces of abstract vector spaces

Last Time: We defined abstract vector spaces: $(\mathbb{V}, +, \cdot)$

- We need a set $\boxed{\mathbb{V}}$. We call each element of $\mathbb{V}$ a vector $\vec{v}$
- We need to have two operations on $\mathbb{V}$:

① **Addition**: $+ : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{V}$ (add two vectors to get a new vector)

$$(\vec{u}, \vec{v}) \longmapsto \vec{u} + \vec{v}$$

② **Scalar Multiplication**: $\cdot \; \mathbb{R} \times \mathbb{V} \longrightarrow \mathbb{V}$ ($\alpha\vec{v}$ is a new vector)

$$(\underset{\text{scalar}}{\alpha}, \underset{\text{vector}}{\vec{v}}) \longrightarrow \alpha\vec{v}$$

- We need $+$ & $\cdot$ to have nice properties (10 total, · + Commutative, · Assoc, Distrib, · $\vec{0}$ = neutral elem, · Additive inverse)

Examples: ① $\mathbb{R}^n$ & subspaces $\mathbb{V}$ of $\mathbb{R}^n$ (usual $+$ & $\cdot$)

② $\mathrm{Mat}_{n\times m}(\mathbb{R})$ = $n \times m$ matrices with real entries (———)

③ Polynomials $\mathcal{P}_n = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n : a_0, \ldots, a_n \text{ in } \mathbb{R}\}$

($+$ & $\cdot$ = coeff-by-coeff)

Useful Properties

① Cancellation Laws:

(1) If $\vec{u}+\vec{v}=\vec{u}+\vec{w}$, then $\vec{v}=\vec{w}$

(2) If $\vec{v}+\vec{u}=\vec{w}+\vec{u}$ then $\vec{v}=\vec{w}$

② Theorem:

(1) The zero vector $\vec{0}$ (= neutral element) is unique

(2) The additive inverse $-\vec{v}$ for $\vec{v}$ is unique & $-\vec{v}=(-1)\cdot\vec{v}$.

(3) $0.\vec{v}=\vec{0}$ for all $\vec{v}$.

(4) $\alpha\cdot\vec{0}=\vec{0}$ for all α scalar.

(5) If $\alpha\cdot\vec{v}=\vec{0}$ then either $\alpha=0$ or $\vec{v}=\vec{0}$.

(1) The zero vector $\vec{0}$ (= neutral element) is unique.

(2) The additive inverse $-\vec{v}$ for $\vec{v}$ is unique & $-\vec{v} = (-1)\cdot\vec{v}$.

(3) $0 \cdot \vec{v} = \vec{0}$ for all $\vec{v}$.

(4) $\alpha \cdot \vec{0} = \vec{0}$ for all α scalar.

Subspaces

Next step: Find subsets $\mathbb{W}$ of a vector space $\mathbb{V}$ that are also a vector space (with the same operations $+$ & $\cdot$) This means we need to check the 10 properties for $\mathbb{W}$. To be a vector space

Q: What happened for $\mathbb{V} = \mathbb{R}^n$?

A: Most of the 10 properties for $\mathbb{W}$ where inherited from those of $\mathbb{V}$ except

Closure Prop (C1) $\vec{u}, \vec{v}$ in $\mathbb{W}$, then $\vec{u} + \vec{v}$ must be in $\mathbb{W}$

(C2) $\vec{u}$ in $\mathbb{W}$ & α scalar, then $\alpha \cdot \vec{u}$ must be in $\mathbb{W}$

Neutral Elem (A3) $\vec{0}$ must be in $\mathbb{W}$ (Here $\vec{0}$ = Neutral elem in $\mathbb{V}$, which we know is unique)

Theorem 2: A subset $\mathbb{W}$ of $\mathbb{V}$ is a subspace of $\mathbb{V}$ if and only if

(S1) $\vec{0}$ lies in $\mathbb{W}$. ($\vec{0}$ = the unique neutral element in $\mathbb{V}$)

(S2) $\vec{x} + \vec{y}$ lies in $\mathbb{W}$ whenever $\vec{x}, \vec{y}$ are in $\mathbb{W}$

(S3) $\alpha \vec{x}$ ______________ $\vec{x}$ is in $\mathbb{W}$ and α is any scalar

Examples

1. $\boxed{C_{[0,1]}} = \{ f : [0,1] \longrightarrow \mathbb{R}$ continuous $\}$ with pointed $+$ & $\cdot$

② $V = P_2$, $W = \{ P(x) \text{ in } P_2 : P'(0) = 0 \}$.

③ $V = P_2$ $W = \{ P(x) \text{ in } P_2 : P(0) = 1 \}$

④ $\mathbb{V} = \mathrm{Mat}_{2\times 3}(\mathbb{R}) =$ 2x3 matrices is a vector space $\left(\mathbb{O} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$

$\mathbb{W} = \left\{ \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \end{pmatrix} : a_{11}, a_{13}, a_{22} \text{ in } \mathbb{R} \right\}$

⑤ $\mathbb{V} = \mathrm{Mat}_{2\times 3}(\mathbb{R})$

$\mathbb{W} = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} : a_{11}a_{22} - a_{12}a_{21} = 0 \right\}$

$\boxed{\text{Spanning Sets}}$

Fix $\mathbb{V}$ = vector space

We use the same definition / constructions as in $\mathbb{R}^n$

Def: A vector $\vec{v}$ in $\mathbb{V}$ is a linear combination of $\vec{v}_1, \dots, \vec{v}_r$ in $\mathbb{V}$ if $\vec{v} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_r \vec{v}_r$ for some $\alpha_1, \dots, \alpha_r$.

Write $\mathbb{W} = \boxed{Sp(\vec{v}_1, \dots, \vec{v}_r)}$ for the set of all linear comb of $\vec{v}_1, \dots, \vec{v}_r$

Def: A set of vectors $\vec{v}_1, \dots, \vec{v}_r$ **spans** $\mathbb{V}$ if $\mathbb{V} = Sp(\vec{v}_1, \dots, \vec{v}_r)$

Examples: 1. $Mat_{2\times 3}(\mathbb{R})$

2. $\mathbb{W} = \left\{ \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \end{bmatrix} : a_{11}, a_{13}, a_{22} \text{ in } \mathbb{R} \right\}$

Theorem 3: If $\mathbb{V}$ is a vector space & $\vec{v}_1, \ldots, \vec{v}_r$ are vectors in $\mathbb{V}$, then $\mathbb{W} = Sp(\vec{v}_1, \ldots, \vec{v}_r)$ is a subspace of $\mathbb{V}$.