Lecture 22: §5.3 Subspaces of abstract vector spaces

Last Time: We defined abstract vector spaces: $(\mathbb{V}, +, \cdot)$

- We need a set $\boxed{\mathbb{V}}$. We call each element of $\mathbb{V}$ a vector $\vec{v}$
- We need to have two operations on $\mathbb{V}$:

① Addition: $+ : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{V}$ (add two vectors to get a new vector)

$$(\vec{u}, \vec{v}) \longmapsto \vec{u} + \vec{v}$$

② Scalar Multiplication: $\cdot \;\; \mathbb{R} \times \mathbb{V} \longrightarrow \mathbb{V}$ ($\alpha\vec{v}$ is a new vector)

$$(\underset{\text{scalar}}{\alpha}, \underset{\text{vector}}{\vec{v}}) \longrightarrow \alpha \vec{v}$$

- We need $+$ & $\cdot$ to have nice properties (10 total, · Assoc, Distrib, · $\vec{0}$ = neutral elem, · Additive inverse)

Examples:
① $\mathbb{R}^n$ & subspaces $\mathbb{V}$ of $\mathbb{R}^n$ (usual + & ·)
② $\mathrm{Mat}_{n\times m}(\mathbb{R})$ = $n\times m$ matrices with real entries (———)
③ Polynomials $\mathcal{P}_n = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n : a_0, \ldots, a_n \text{ in } \mathbb{R}\}$
(+ & · = coeff-by-coeff)

Useful Properties

① Cancellation Laws:

(1) If $\vec{u}+\vec{v}=\vec{u}+\vec{w}$, then $\vec{v}=\vec{w}$

Why? Take $\vec{u}'=-\vec{u}$ (additive inverse)

$$\left.\begin{array}{l} \vec{u}'+(\vec{u}+\vec{v}) \underset{\text{Assoc}}{=} (\vec{u}'+\vec{u})+\vec{v} = \vec{0}+\vec{v} = \vec{v} \\ \quad \| \qquad\qquad\qquad\qquad\qquad\qquad \vec{0}\text{ neutral elem} \\ \vec{u}'+(\vec{u}+\vec{w}) = (\vec{u}'+\vec{u})+\vec{w} = \vec{0}+\vec{w} = \vec{w} \end{array}\right\} \text{gives } \vec{v}=\vec{w}$$

(2) If $\vec{v}+\vec{u}=\vec{w}+\vec{u}$ then $\vec{v}=\vec{w}$ (Use (1) + Commutative prop)

② Theorem:

(1) The zero vector $\vec{0}$ (= neutral element) is unique

(2) The additive inverse $-\vec{v}$ for $\vec{v}$ is unique & $-\vec{v}=(-1)\cdot\vec{v}$.

(3) $0\cdot\vec{v}=\vec{0}$ for all $\vec{v}$.

(4) $\alpha\cdot\vec{0}=\vec{0}$ for all α scalar.

(5) If $\alpha\cdot\vec{v}=\vec{0}$ then either $\alpha=0$ or $\vec{v}=\vec{0}$.

For (5) If $\alpha\neq 0$: $\vec{v}=1\cdot\vec{v}=\left(\frac{1}{\alpha}\alpha\right)\vec{v}\overset{\text{Assoc}}{=}\frac{1}{\alpha}(\alpha\vec{v})=\frac{1}{\alpha}\vec{0}\overset{(4)}{=}\vec{0}$

(1) The zero vector $\vec{0}$ (= neutral element) is unique.

If we have 2 neutral elements $\vec{0}$ & $\vec{0}'$, then $\vec{0} \underset{\vec{0}' \text{ Neutral}}{=} \vec{0} + \vec{0}' \underset{\vec{0} \text{ Neutral}}{=} \vec{0}'$

(2) The additive inverse $-\vec{v}$ for $\vec{v}$ is unique & $-\vec{v} = (-1)\cdot\vec{v}$.

- If we have 2 inverses $\vec{u}, \vec{u}'$ $\left(\vec{u} + \vec{v} = \vec{u}' + \vec{v} = \vec{0}\right)$, then

$$\vec{u} = \vec{u} + \vec{0} = \vec{u} + (\vec{u}' + \vec{v}) \underset{\text{Comm}}{=} \vec{u} + (\vec{v} + \vec{u}') \underset{\text{Assoc}}{=} (\vec{u} + \vec{v}) + \vec{u}' = \vec{0} + \vec{u}' = \vec{u}'$$

- $(-1)\vec{v}$ satisfies $\vec{v} + (-1)\vec{v} = 1\cdot\vec{v} + (-1)\vec{v} \underset{\text{Dist}}{=} (1-1)\vec{v} = 0\cdot\vec{v} \underset{(3)}{=} \vec{0}$

So $(-1)\vec{v}$ works as an inverse.

(3) $0\cdot\vec{v} = \vec{0}$ for all $\vec{v}$.

$$\vec{0} + 0\cdot\vec{v} \underset{\vec{0} \text{ Neutral}}{=} 0\cdot\vec{v} = (0+0)\cdot\vec{v} \underset{\text{Distrib}}{=} 0\cdot\vec{v} + 0\cdot\vec{v} \underset{\text{Cancellation}}{\rightsquigarrow} \vec{0} = 0\cdot\vec{v}$$

(4) $\alpha\cdot\vec{0} = \vec{0}$ for all α scalar.

$$\vec{0} + \alpha\cdot\vec{0} = \alpha\cdot\vec{0} = \alpha\cdot(\vec{0} + \vec{0}) \underset{\text{Distrib}}{=} \alpha\cdot\vec{0} + \alpha\cdot\vec{0} \underset{\text{Cancellation}}{\rightsquigarrow} \vec{0} = \alpha\cdot\vec{0}$$

Subspaces

Next step: Find subsets $\mathbb{W}$ of a vector space $\mathbb{V}$ that are also a vector space (with the same operations $+$ & $\cdot$) This means we need to check the 10 properties for $\mathbb{W}$ to be a vector space

Q: What happened for $\mathbb{V} = \mathbb{R}^n$?

A: Most of the 10 properties for $\mathbb{W}$ where inherited from those of $\mathbb{V}$ except

Closure Prop (C1) $\vec{u}, \vec{v}$ in $\mathbb{W}$, then $\vec{u}+\vec{v}$ must be in $\mathbb{W}$

________ (C2) $\vec{u}$ in $\mathbb{W}$ & α scalar, then $\alpha \cdot \vec{u}$ must be in $\mathbb{W}$

Neutral Elem (A3) $\vec{0}$ must be in $\mathbb{W}$ (Here $\vec{0}$ = Neutral elem in $\mathbb{V}$, which we know is unique)

Theorem 2: A subset $\mathbb{W}$ of $\mathbb{V}$ is a subspace of $\mathbb{V}$ if and only if

(S1) $\vec{0}$ lies in $\mathbb{W}$. ($\vec{0}$ = the unique neutral element in $\mathbb{V}$)

(S2) $\vec{x}+\vec{y}$ lies in $\mathbb{W}$ whenever $\vec{x}, \vec{y}$ are in $\mathbb{W}$

(S3) $\alpha \vec{x}$ ________________ $\vec{x}$ is in $\mathbb{W}$ and α is any scalar

Examples

1. $\boxed{C_{[0,1]}} = \{ f : [0,1] \to \mathbb{R}$ continuous $\}$ with pointed $+$ & $\cdot$

- f, g in $C[0,1]$ $\rightsquigarrow$ $f+g$ is a function with $(f+g)_{(x)} = f_{(x)} + g_{(x)}$
  (continuous!) numbers!
- f in $C[0,1]$, α scalar $\rightsquigarrow$ $\alpha \cdot f$ is a function with $(\alpha \cdot f)_{(x)} = \alpha f_{(x)}$
  (continuous!)
- $\vec{0}$ = constant zero function ($\vec{0}_{(x)} = 0$ for all x in $[0,1]$; so cont.)

All 10 properties for vector spaces are satisfied, so $C[0,1]$ is a vector space

Q: Subspaces? $\mathcal{P}_2 = \{ a + bx + cx^2 : a, b, c \}$ View polynomials as functions!

Eg $P_{(x)} = 1 - 4x + 5x^2$, $P_{(1)} = 1 - 4 \cdot 1 + 5 \cdot 1^2 = 1 - 4 + 5 = 2$, etc)

Polynomials are continuous, $\vec{0} = 0 + 0 \cdot x + 0 \cdot x^2$ is constant zero function

$+$ & $\cdot$ defined coeff-by-coefficient is the pointed $+$ & $\cdot$ we defined above

So (S1), (S2) & (S3) hold for $\mathcal{P}_2$. <u>Same</u> will be true for $\mathcal{P}_n = \{ a_0 + a_1 x + \dots + a_n x^n$: $a_0, \dots, a_n$ in $\mathbb{R} \}$

(2) $\mathbb{V} = \mathcal{P}_2$, $\mathbb{W} = \{ P(x) \text{ in } \mathcal{P}_2 : P'(0) = 0 \}$ is a subspace of $\mathcal{P}_2$.

(S1) $\vec{0} = 0 + 0\cdot x + 0x^2$ in $\mathbb{W}$, then $(\vec{0})'_{(x)} = 0$ also constant, so $(\vec{0})'_{(0)} = 0$.
& $\vec{0}$ is in $\mathbb{W}$.

(S2) P, Q with $P'(0) = Q'(0) = 0$, then $(P+Q)'_{(0)} = (P'+Q')_{(0)} = 0+0=0$
so $P+Q$ lies in $\mathbb{W}$

(S3) P in $\mathbb{W}$, α scalar, then $(\alpha P)'_{(0)} = (\alpha P')_{(0)} = \alpha P'_{(0)} = \alpha \cdot 0 = 0$.
so αP lies in $\mathbb{W}$

Alternative:

$$\left.\begin{array}{l} \underset{\text{in } \mathbb{W}}{P(x) = a + bx + cx^2} \rightsquigarrow P'(x) = b + 2cx \\ \qquad\qquad\qquad\qquad 0 = P'(0) = b \end{array}\right\} \rightsquigarrow P(x) = a + cx^2.$$

So $\mathbb{W} = \{ a + cx^2 : a, c \text{ in } \mathbb{R} \}$ (no x term) $= Sp(1, x^2)$

This is clearly closed under $+$ & $\cdot$ & $\vec{0} = 0 + 0\cdot x^2$. is in $\mathbb{W}$.

(3) $\mathbb{V} = \mathcal{P}_2$ $\mathbb{W} = \{ P(x) \text{ in } \mathcal{P}_2 : P(0) = 1 \}$ is <u>not</u> a subspace

$\vec{0}$ is not in $\mathbb{W}$, so (S1) fails.

④ $\mathbb{V} = \mathrm{Mat}_{2\times 3}(\mathbb{R})$ = 2×3 matrices is a vector space $\left(\mathbb{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$

$\mathbb{W} = \left\{ \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \end{pmatrix} : a_{11}, a_{13}, a_{22} \text{ in } \mathbb{R} \right\}$ is a subspace of $\mathbb{V}$

(S1) $\mathbb{0}$ in $\mathbb{W}$ (take $a_{11} = a_{13} = a_{22} = 0$)

(S2) $\left.\begin{array}{l} A = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \end{pmatrix} \\ B = \begin{pmatrix} b_{11} & 0 & b_{13} \\ 0 & b_{22} & 0 \end{pmatrix} \end{array}\right\} \rightsquigarrow (A+B) = \begin{pmatrix} a_{11}+b_{11} & 0+0 & a_{13}+b_{13} \\ 0+0 & a_{22}+b_{22} & 0+0 \end{pmatrix}$

$= \begin{pmatrix} a_{11}+b_{11} & 0 & a_{13}+b_{13} \\ 0 & a_{22}+b_{22} & 0 \end{pmatrix}$ is in $\mathbb{W}$

(S3) $\left.\begin{array}{l} A = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \end{pmatrix} \\ \alpha \text{ scalar} \end{array}\right\} \rightsquigarrow \alpha \cdot A = \begin{pmatrix} \alpha a_{11} & 0 & \alpha a_{13} \\ 0 & \alpha a_{22} & 0 \end{pmatrix}$ is in $\mathbb{W}$.

⑤ $\mathbb{V} = \mathrm{Mat}_{2\times 3}(\mathbb{R})$ $\mathbb{W} = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} : a_{11}a_{22} - a_{12}a_{21} = 0 \right\}$

is NOT a subspace

(S2) fails $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ in $\mathbb{W}$

$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ in $\mathbb{W}$ but $A+B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ is NOT in $\mathbb{W}$!

Spanning Sets

Fix $\mathbb{V}$ = vector space

We use the same definition / constructions as in $\mathbb{R}^n$

Def: A vector $\vec{v}$ in $\mathbb{V}$ is a linear combination of $\vec{v}_1, \dots, \vec{v}_r$ in $\mathbb{V}$ if $\vec{v} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_r \vec{v}_r$ for some $\alpha_1, \dots, \alpha_r$.

Write $\mathbb{W} = \boxed{Sp(\vec{v}_1, \dots, \vec{v}_r)}$ for the set of all linear comb of $\vec{v}_1, \dots, \vec{v}_r$

$$= \{ \alpha_1 \vec{v}_1 + \dots + \alpha_r \vec{v}_r : \alpha_1, \dots, \alpha_r \text{ in } \mathbb{R} \}$$

Example: $\mathcal{P}_2 = \{ a + bx + cx^2 : a, b, c \text{ in } \mathbb{R} \} = Sp(1, x, x^2)$

Def: A set of vectors $\vec{v}_1, \dots, \vec{v}_r$ spans $\mathbb{V}$ if $\mathbb{V} = Sp(\vec{v}_1, \dots, \vec{v}_r)$

⚠ NOT all vector spaces have finite spanning sets

Ex: $C[0,1]$ has no spanning set. We'll need $1, x, x^2, x^3, \dots$ and more!

Examples: ① $Mat_{2\times 3}(\mathbb{R}) = Sp\left(\underset{E_{11}}{\begin{bmatrix} 1&0&0\\0&0&0\end{bmatrix}}, \underset{E_{12}}{\begin{bmatrix} 0&1&0\\0&0&0\end{bmatrix}}, \underset{E_{13}}{\begin{bmatrix} 0&0&1\\0&0&0\end{bmatrix}}, \underset{E_{21}}{\begin{bmatrix} 0&0&0\\1&0&0\end{bmatrix}}, \underset{E_{22}}{\begin{bmatrix} 0&0&0\\0&1&0\end{bmatrix}}, \underset{E_{23}}{\begin{bmatrix} 0&0&0\\0&0&1\end{bmatrix}} \right)$

Why? $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23}\end{bmatrix} = a_{11}E_{11} + a_{12}E_{12} + a_{13}E_{13} + a_{21}E_{21} + a_{22}E_{22} + a_{23}E_{23}$

scalars

② $\mathbb{W} = \left\{ \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \end{bmatrix} : a_{11}, a_{13}, a_{22} \text{ in } \mathbb{R} \right\} = Sp(E_{11}, E_{13}, E_{22})$.

Theorem 3: If $\mathbb{V}$ is a vector space & $\vec{v}_1, \dots, \vec{v}_r$ are vectors in $\mathbb{V}$, then $\mathbb{W} = Sp(\vec{v}_1, \dots, \vec{v}_r)$ is a subspace of $\mathbb{V}$.

Proof: (S1) $\vec{0} = \underbrace{\vec{0} + \vec{0} + \cdots + \vec{0}}_{r \text{ times}} \underset{\text{Thm 1}}{=} 0\cdot\vec{v}_1 + 0\cdot\vec{v}_2 + \cdots + 0\vec{v}_r$

(S2)
$$\begin{aligned} \vec{v} &= \alpha_1\vec{v}_1 + \cdots + \alpha_r\vec{v}_r \quad \text{in } \mathbb{W} \\ + \quad \vec{w} &= \beta_1\vec{v}_1 + \cdots + \beta_r\vec{v}_r \quad \text{in } \mathbb{W} \\ \hline \vec{v}+\vec{w} &= (\alpha_1+\beta_1)\vec{v}_1 + \cdots + (\alpha_r+\beta_r)\vec{v}_r \quad \text{is also in } \mathbb{W}. \ (*) \end{aligned}$$

In (*) we secretly use Commutative, Distrib & Assoc Properties in $\mathbb{V}$.

(S3) $\alpha\cdot\vec{w} = (\alpha\beta_1)\vec{v}_1 + \cdots + (\alpha\beta_r)\vec{v}_r$ so also in $\mathbb{W}$ (*)