Lecture 23: §5.4 Linear Independence & Bases

Last Time: Defined $\mathbb{W}$ subspace of an abstract vector space $\mathbb{V}$: Same $+$ & $\cdot$ &

(S1) $\vec{0}$ in $\mathbb{W}$ ($\vec{0}$ = (unique) neutral element for $+$ in $\mathbb{V}$)

(S2) If $\vec{u}, \vec{v}$ in $\mathbb{W}$, then $\vec{u}+\vec{v}$ in $\mathbb{W}$

(S3) If $\vec{u}$ in $\mathbb{W}$ & α scalar, then $\alpha\cdot\vec{u}$ also in $\mathbb{W}$

Examples: • $\mathbb{R}^n$ & usual subspaces

• $\mathbb{V} = C_{[0,1]} = \{ f: [0,1] \to \mathbb{R} \text{ cont. } f \}$ & $\mathbb{W}_1 = \mathcal{P}_2$

$\mathbb{W}_2 = \{ f \text{ in } C_{[0,1]} : f'(0) = 0 \}$

Main Example: $\mathbb{W} = \mathrm{Sp}(\vec{v}_1, \dots \vec{v}_r) = \{ \alpha_1 \vec{v}_1 + \dots + \alpha_r \vec{v}_r : \alpha_1, \dots \alpha_r \text{ in } \mathbb{R} \}$

= all linear comb. of $\vec{v}_1, \dots, \vec{v}_r$.

⚠ $C_{[0,1]}$ has NO finite spanning sets.

Examples $\mathcal{P}_2 = \{ a + bx + cx^2 : a, b, c \text{ in } \mathbb{R} \} = \mathrm{Sp}(1, x, x^2)$

$\mathrm{Mat}_{2\times 3} = \mathrm{Sp}(E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23})$ $\quad E_{ij} = \begin{cases} 1 & \text{in } (i,j) \text{ spot} \\ 0 & \text{elsewhere} \end{cases}$

$\mathbb{W} = \left\{ \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \end{pmatrix} : a_{11}, a_{13}, a_{22} \text{ in } \mathbb{R} \right\} = \mathrm{Sp}(E_{11}, E_{13}, E_{22})$

Example: $\mathbb{W} = \{ A \text{ in } \mathrm{Mat}_{3\times 3} : A^T = A \}$ (symmetric 3×3 matrices)

Linear Independence

We use the same definition as the one from $\mathbb{R}^n$. $\rightsquigarrow$ same methods to check l.i/l.d.

Def: Fix a vector space $\mathbb{V}$ & vectors $\vec{v}_1, \ldots, \vec{v}_r$ in $\mathbb{V}$. We write

$$(*) \qquad \vec{0} = \alpha_1 \cdot \vec{v}_1 + \cdots + \alpha_r \cdot \vec{v}_r \qquad \text{for } \alpha_1, \ldots, \alpha_r \text{ unknowns}$$

- We say $\{\vec{v}_1, \ldots, \vec{v}_r\}$ is a linearly independent set if the ONLY solution to $(*)$ is $\alpha_1 = \alpha_2 = \cdots = \alpha_r = 0$
- Otherwise, we say $\{\vec{v}_1, \ldots, \vec{v}_r\}$ is linearly dependent

Prop: If we have a dependency relation $\alpha_1 \vec{v}_1 + \cdots + \alpha_r \vec{v}_r = \vec{0}$ with $\alpha_i \neq 0$, then $\vec{v}_i$ is in $Sp(\vec{v}_1, \ldots \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_r)$

METHOD 1 : Subsets of $Mat_{m\times n}$

Example 1 : $\{E_{11}, E_{12}+E_{21}, E_{13}+E_{31}, E_{22}, E_{23}+E_{32}, E_{33}\}$ in $Mat_{3\times 3}$ is l.i

Example 2 $\left\{ \vec{v}_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 3 & 0 & 5 \\ 0 & 3 & 0 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}$ is ld

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbb{O} = a \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + b \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} + c \begin{bmatrix} 3 & 0 & 5 \\ 0 & 3 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Method 2: For spaces involving functions/polynomials

Example 3: $\{1, x, x^2\}$ is li in $\mathcal{P}_2$ Write: $0 = a + bx + cx^2$

- Evaluate at convenient values of x to get a linear system in a, b, c.

Example 4: $\{1, (x+1)^2, (x-1)^2, x^2\}$ ld.

Write $0 = a + b(x+1)^2 + c(x-1)^2 + dx^2$

Option 1: Expand polynomials & regroup by monomials. Then, use $\{1, x, x^2\}$ is li

Option 2: Write ① $= a + b(x+1)^2 + c(x-1)^2 + dx^2$ & evaluate at ≥ 4 values for x

(vanishing of polys)

Option 3: Write $\textcircled{1} = a + b(x+1)^2 + c(x-1)^2 + d x^2$, & take derivatives up to order 2.

(2 = largest degree)

Bases for abstract vector spaces

Def: Fix V a vector space. A set $B = \{\vec{v}_1, \dots, \vec{v}_r\}$ is a basis for V if

(1) B is a spanning set for V

&

(2) B is l.i.

Equivalently: B is a minimal spanning set. (spans + removing a vector doesn't span)

Examples: ① $Mat_{2\times 3}$ has basis

② $Mat_{m\times n}$ ______

③ $P_d = \{a_0 + a_1 x + \cdots + a_d x^d\}$ has basis
$a_0, \dots, a_d \in \mathbb{R}$

Key fact: All basis of a vector space V have the same number of elements. We define this number as the dimension of V.

⚠ Not every vector space has a finite basis (Eg $C[0,1]$ does NOT)