Lecture 23: §5.4 Linear Independence & Bases

Last Time: Defined $\mathbb{W}$ subspace of an abstract vector space $\mathbb{V}$: Same $+$ & $\cdot$ &

(S1) $\vec{0}$ in $\mathbb{W}$ ($\vec{0}$ = (unique) neutral element for $+$ in $\mathbb{V}$)

(S2) If $\vec{u}, \vec{v}$ in $\mathbb{W}$, then $\vec{u}+\vec{v}$ in $\mathbb{W}$

(S3) If $\vec{u}$ in $\mathbb{W}$ & α scalar, then $\alpha \cdot \vec{u}$ also in $\mathbb{W}$

Examples:
- $\mathbb{R}^n$ & usual subspaces
- $\mathbb{V} = C[0,1] = \{ f : [0,1] \to \mathbb{R} \text{ cont. } f \}$ & $\mathbb{W}_1 = \mathcal{P}_2$

  $\mathbb{W}_2 = \{ f \text{ in } C[0,1] : f'(0) = 0 \}$

Main Example: $\mathbb{W} = \mathrm{Sp}(\vec{v}_1, \ldots, \vec{v}_r) = \{ \alpha_1 \vec{v}_1 + \cdots + \alpha_r \vec{v}_r : \alpha_1, \ldots, \alpha_r \text{ in } \mathbb{R} \}$

= all linear comb. of $\vec{v}_1, \ldots, \vec{v}_r$.

⚠ $C[0,1]$ has no finite spanning sets.

Examples

$\mathcal{P}_2 = \{ a + bx + cx^2 : a, b, c \text{ in } \mathbb{R} \} = \mathrm{Sp}(1, x, x^2)$

$\mathrm{Mat}_{2\times 3} = \mathrm{Sp}(E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23})$ $\quad E_{ij} = \begin{cases} 1 & \text{in } (i,j) \text{ spot} \\ 0 & \text{elsewhere} \end{cases}$

$\mathbb{W} = \left\{ \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \end{pmatrix} : a_{11}, a_{13}, a_{22} \text{ in } \mathbb{R} \right\} = \mathrm{Sp}(E_{11}, E_{13}, E_{22})$

Example: $W = \{ A \text{ in } Mat_{3\times3} : A^T = A \}$ (symmetric 3×3 matrices)

- It is a subspace because $\mathbb{0}^T = \mathbb{0}$ & $\begin{cases} (A+B)^T = A^T + B^T \\ (\alpha A)^T = \alpha A^T \end{cases}$ (transpose rules)

Q: Spanning Set?

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{22} & a_{33} \end{pmatrix} \rightsquigarrow A^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

same diagonal entries.

$A = A^T$ gives
$$\begin{aligned} a_{12} &= a_{21} \\ a_{13} &= a_{31} \\ a_{23} &= a_{32} \end{aligned} \rightsquigarrow A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

$$\rightsquigarrow A = a_{11}E_{11} + a_{12}E_{12} + a_{13}E_{13} + a_{21}E_{21} + a_{22}E_{22} + a_{23}E_{23} + a_{31}E_{31} + a_{32}E_{32} + a_{33}E_{33}$$

We regroup by coefficients (using matching colored terms)

$$A = a_{11}E_{11} + a_{12}(E_{12} + E_{21}) + a_{13}(E_{13} + E_{31}) + a_{22}E_{22} + a_{23}(E_{23} + E_{32}) + a_{33}E_{33}$$

$a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33}$ have <u>no</u> restriction

Conclude: $W = Sp(E_{11}, E_{12} + E_{21}, E_{13} + E_{31}, E_{22}, E_{23} + E_{32}, E_{33})$

Linear Independence

We use the same definition as the one from $\mathbb{R}^n$. $\rightsquigarrow$ same methods to check l.i/l.d.

Def: Fix a vector space V & vectors $\vec{v}_1, \dots, \vec{v}_r$ in V. We write

$$(*) \qquad \vec{0} = \alpha_1 \cdot \vec{v}_1 + \dots + \alpha_r \cdot \vec{v}_r \qquad \text{for } \alpha_1, \dots, \alpha_r \text{ unknowns}$$

- We say $\{\vec{v}_1, \dots, \vec{v}_r\}$ is a linearly independent set if the ONLY solution to $(*)$ is $\alpha_1 = \alpha_2 = \dots = \alpha_r = 0$
- Otherwise, we say $\{\vec{v}_1, \dots, \vec{v}_r\}$ is linearly dependent

Prop: If we have a dependency relation $\alpha_1 \vec{v}_1 + \dots + \alpha_r \vec{v}_r = \vec{0}$ with $\alpha_i \neq 0$, then $\vec{v}_i$ is in $\mathrm{Sp}(\vec{v}_1, \dots \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_r)$

Why?

$$\alpha_i \vec{v}_i = -\alpha_1 \vec{v}_1 - \dots - \alpha_{i-1} \vec{v}_{i-1} - \alpha_{i+1} \vec{v}_{i+1} - \dots - \alpha_r \vec{v}_r$$

$$\rightsquigarrow \vec{v}_i = \boxed{\frac{-\alpha_1}{\alpha_i}} \vec{v}_1 - \dots \boxed{-\frac{\alpha_{i-1}}{\alpha_i}} \vec{v}_{i-1} \boxed{-\frac{\alpha_{i+1}}{\alpha_i}} \vec{v}_{i+1} - \dots \boxed{\frac{-\alpha_r}{\alpha_i}} \vec{v}_r$$

scalars

METHOD 1: Subsets of $\mathrm{Mat}_{m\times n}$

Example 1: $\{E_{11}, E_{12}+E_{21}, E_{13}+E_{31}, E_{22}, E_{23}+E_{32}, E_{33}\}$ in $\mathrm{Mat}_{3\times 3}$ is l.i

Why? Write

$$0 = \begin{bmatrix} 0&0&0\\0&0&0\\0&0&0\end{bmatrix} = a_{11}E_{11} + a_{12}(E_{12}+E_{21}) + a_{13}(E_{13}+E_{31}) + a_{22}E_{22} + a_{23}(E_{23}+E_{32}) + a_{33}E_{33}. \quad (*)$$

$$\begin{bmatrix} 0&0&0\\0&0&0\\0&0&0\end{bmatrix} = a_{11}\begin{bmatrix} 1&0&0\\0&0&0\\0&0&0\end{bmatrix} + a_{12}\begin{bmatrix} 0&1&0\\1&0&0\\0&0&0\end{bmatrix} + a_{13}\begin{bmatrix} 0&0&1\\0&0&0\\1&0&0\end{bmatrix} + a_{22}\begin{bmatrix} 0&0&0\\0&1&0\\0&0&0\end{bmatrix} + a_{23}\begin{bmatrix} 0&0&0\\0&0&1\\0&1&0\end{bmatrix} + a_{33}\begin{bmatrix} 0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

$$\begin{bmatrix} 0&0&0\\0&0&0\\0&0&0\end{bmatrix} = \begin{bmatrix} a_{11}&a_{12}&a_{13}\\a_{12}&a_{22}&a_{23}\\a_{13}&a_{23}&a_{33}\end{bmatrix}$$

$\rightsquigarrow$ Equate the entries to get 6 equations (9 total but 3 repeated)

$0 = a_{11}$, $0 = a_{12}$, $0 = a_{13}$, $0 = a_{22}$, $0 = a_{23}$ & $0 = a_{33}$

Conclude: Only solution to $(*)$ is the trivial one (all $a_{ij} = 0$)

So li by definition.

Example ② $\left\{ \vec{v}_1 = \begin{bmatrix} 1&0&1 \\ 0&1&0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1&0&2 \\ 0&1&0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 3&0&5 \\ 0&3&0 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 0&0&1 \\ 0&0&0 \end{bmatrix} \right\}$ is ld

$$\begin{bmatrix} 0&0&0 \\ 0&0&0 \end{bmatrix} = \vec{0} = a \begin{bmatrix} 1&0&1 \\ 0&1&0 \end{bmatrix} + b \begin{bmatrix} 1&0&2 \\ 0&1&0 \end{bmatrix} + c \begin{bmatrix} 3&0&5 \\ 0&3&0 \end{bmatrix} + d \begin{bmatrix} 0&0&1 \\ 0&0&0 \end{bmatrix}$$

$$= \begin{bmatrix} a+b+3c & 0 & a+2b+5c+d \\ 0 & a+b+3c & 0 \end{bmatrix}$$

$\rightsquigarrow$ We get 3 equations (all entries $=0$)

$$\begin{cases} a+b+3c = 0 \\ a+2b+5c+d = 0 \\ a+b+3c = 0 \end{cases}$$ repeated!

2 equations
4 unknowns $\rightsquigarrow$ we have a nontrivial solution because $(2<4)$ we have inf. many!

Solution? $\begin{bmatrix} 1&1&3&0 \\ 1&2&5&1 \end{bmatrix} \xrightarrow[R_2 \to R_2 - R_1]{} \begin{bmatrix} 1&1&3&0 \\ 0&1&2&1 \end{bmatrix} \xrightarrow[R_1 \to R_1 - R_2]{} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1&0&1&-1 \\ 0&1&2&1 \end{bmatrix} \end{array}$ REF

a, b dep
c, d indep

$$\begin{cases} a = -c+d \\ b = -2c-d \end{cases} \rightsquigarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -c+d \\ -2c-d \\ c \\ d \end{bmatrix} = c \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

(I) (II)

$\rightsquigarrow$ 2 building blocks for Relns:

(I) $-\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3 = \vec{0}$ & (II) $\vec{v}_1 - \vec{v}_2 + \vec{v}_4 = \vec{0}$

$$\boxed{\begin{array}{c} Sp(v_1, \ldots, v_4) \\ \parallel \\ Sp(v_1, v_2) \end{array}}$$

$\underline{\text{Method 2}}$: For spaces involving functions / polynomials

Example 3: $\{1, x, x^2\}$ is li in $\mathcal{P}_2$ Write: $0 = a + bx + cx^2$

. Evaluate at convenient values of x (at least 3) to get a linear system in a, b, c.

At $x=0$: $0 = a$

At $x=1$: $0 = a+b+c = b+c$

At $x=-1$: $0 = a-b+c = -b+c$

$\left.\begin{matrix} 0 = b+c \\ 0 = -b+c \end{matrix}\right\} b=c=0$ so $a=b=c=0$

Example 4: $\{1, (x+1)^2, (x-1)^2, x^2\}$ ld.

Write $0 = a + b(x+1)^2 + c(x-1)^2 + dx^2$

Option 1: Expand polynomials & regroup by monomials. Then, use $\{1, x, x^2\}$ is li

$$0 = a + b(x^2+2x+1) + c(x^2-2x+1) + dx^2$$

$$0 = (a+b+c) + (2b-2c)x + (b+c+d)x^2$$

$$\rightsquigarrow \begin{cases} a+b+c=0 \\ 2b-2c=0 \\ b+c+d=0 \end{cases} \quad \begin{matrix} \rightsquigarrow a+2b=0 \rightsquigarrow a=-2b \\ \rightsquigarrow b=c \\ \rightsquigarrow 2b+d=0 \rightsquigarrow d=-2b \end{matrix} \rightsquigarrow \boxed{\begin{matrix} 0 = -2 + (x-1)^2 + (x-1)^2 - 2x^2 \\ (\text{set } b=1) \end{matrix}}$$

Option 2: Write $\mathbb{0} = a + b(x+1)^2 + c(x-1)^2 + d x^2$ & evaluate at ≥ 4 values for x
(vanishing of polys)

At $x=0$: $\quad 0 = a + 1^2 b + (-1)^2 c + d\, 0^2 = a+b+c$

At $x=1$: $\quad 0 = a + 2^2 b + 0^2 c + d\, 1^2 = a+4b+d$

At $x=-1$: $\quad 0 = a + 0^2 b + (-2)^2 c + d(-1)^2 = a+4c+d$

At $x=2$: $\quad 0 = a + 3^2 b + 1^2 c + d\, 2^2 = a+9b+c+4d$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 4 & 0 & 1 \\ 1 & 0 & 4 & 1 \\ 1 & 9 & 1 & 4 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1}]{} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & -1 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 8 & 0 & 4 \end{bmatrix} \xrightarrow[\substack{R_3 \leftrightarrow R_2 \\ R_2 \to -R_2 \\ R_4 \to \frac{R_4}{4}}]{} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 3 & -1 & 1 \\ 0 & 2 & 0 & 1 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - 3R_2 \\ R_4 \to R_4 - 2R_2}]{} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 8 & 4 \\ 0 & 0 & 6 & 3 \end{bmatrix}$$

$$\xrightarrow[\substack{R_3 \to \frac{R_3}{8} \\ R_4 \to R_4 - 6R_3}]{} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\substack{R_2 \to R_2 + 3R_3 \\ R_1 \to R_1 - R_3 - R_2}]{} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \text{REF} \end{array}$$

$a = d$
$b = \frac{1}{2} d$
$c = \frac{1}{2} d$

Take $d=1$ to get

$$\boxed{\mathbb{0} = 2 - (1+x)^2 - (x-1)^2 + 2x^2}$$

⚠ This is only a potential relation (it's true for $x = 0, 1, -1, 2$). We still need to check it holds for all x (= as functions!) But this is true by Option 1 Soln.

Option 3: Write $0 = a + b(x+1)^2 + c(x-1)^2 + dx^2$, & take derivatives up to order 2.
(2 = largest degree)

- $\frac{d}{dx}$: $\frac{d}{dx}(0) = \frac{d}{dx}\left(a + b(x+1)^2 + c(x-1)^2 + dx^2\right)$

$$0 = 2b(x+1) + 2c(x-1) + 2dx = 2(b+c+d)x + 2(b-c)$$

- $\frac{d^2}{dx^2}$: $0 = 2b + 2c + 2d$

$$\rightsquigarrow \begin{cases} 0 = a + b(x+1)^2 + c(x-1)^2 + dx^2 & (1) \\ 0 = 2(b-c) + 2(b+c+d)x & (2) \\ 0 = 2(b+c+d) & (3) \end{cases}$$

(2) + $\{1, x\}$ li gives $\begin{cases} b - c = 0 \\ b + c + d = 0 \end{cases}$ $\rightsquigarrow$ (3) follows

$\hookrightarrow d = -2b$

Substitute $\boxed{b = c \ \& \ d = -2b}$ in (1).

$$0 = a + b(x+1)^2 + b(x-1)^2 - 2bx^2$$
$$= a + b\left((x+1)^2 + (x-1)^2 - 2x^2\right)$$
$$= a + b(2) \quad \text{so} \quad \boxed{a = -2b}$$

$\rightsquigarrow$ Relation: $\boxed{0 = -2 + (x+1)^2 + (x-1)^2 - 2x^2}$ (set $b = 1$)

Bases for abstract vector spaces

Def: Fix $\mathbb{V}$ a vector space. A set $B=\{\vec{v}_1,\dots,\vec{v}_r\}$ is a **basis** for $\mathbb{V}$ if

(1) B is a spanning set for $\mathbb{V}$

&

(2) B is l.i.

Equivalently: B is a minimal spanning set. (spans + removing a vector doesn't span)

Examples:

1. $\mathrm{Mat}_{2\times 3}$ has basis $\{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}\}$ (6 elem)
2. $\mathrm{Mat}_{m\times n}$ ——— $\{E_{i,j} \;\; i=1,\dots m,\; j=1,\dots n\}$ (m·n elem)
3. $\mathcal{P}_d = \{a_0 + a_1 x + \dots + a_d x^d \mid a_0,\dots,a_d \in \mathbb{R}\}$ has basis $\{1, x, x^2, \dots, x^d\}$ (d+1 elem)

Key fact: All basis of a vector space $\mathbb{V}$ have the same number of elements. We define this number as the **dimension** of $\mathbb{V}$.

⚠ Not every vector space has a finite basis (Eg $C[0,1]$ does NOT)